DISTRIBUTION OF VALUES OF QUADRATIC FORMS AT INTEGRAL POINTS *

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Abstract. The number of lattice points in d-dimensional hyperbolic or elliptic shells $\{m;\ a < Q[m] < b\}$ which are restricted to rescaled and growing domains $r\,\Omega$ is approximated by the volume. An effective error bound of order $o(r^{d-2})$ for this approximation is proved based on Diophantine approximation properties of the quadratic form Q. These results allow to show effective variants of previous non-effective results in the quantitative Oppenheim problem and extend known effective results in dimension $d \geq 9$ to dimension $d \geq 5$. They apply to wide shells when $b-a$ is growing with r and to *positive* forms Q. For indefinite forms they provide explicit bounds for the norm of non-zero integral points m in dimension $d \geq 5$ solving the Diophantine inequality $|Q[m]| < \varepsilon$.

1. Introduction and Results

Let $Q[x]$ denote an indefinite quadratic form in d variables. We say, that the form Q is *rational*, if it is proportional to a form with integer coefficients; otherwise it is called *irrational*. The Oppenheim conjecture, proved by G. Margulis 1986, [Marg89] stated, that if $d \geq 3$ and Q is irrational, then $Q[\mathbb{Z}^d]$ is dense in $\mathbb{R}$. The proof given in 1986 uses a connection, noticed by M. S. Raghunathan between the Oppenheim conjecture and questions concerning closures in $SL(3,\mathbb{R})/SL(3,\mathbb{Z})$ of orbits of certain subgroups of $SL(3,\mathbb{R})$. It is based on the study of minimal invariant sets and the limits of orbits of sequences of points tending to a minimal invariant set.

For a (measurable) set $B \subset \mathbb{R}^d$, vol B denotes the Lebesgue measure of B and $\mathrm{vol}_{\mathbb{Z}} B \stackrel{\text{def}}{=} \#(B \cap \mathbb{Z}^d)$ denotes the number of integer points on B. We define for $a, b \in \mathbb{R}$, with $a < b$ the *hyperbolic shell*

$$E_{a,b} \stackrel{\text{def}}{=} \{x \in \mathbb{R}^d; a < Q[x] < b\}.$$

The Oppenheim conjecture is equivalent to the statement, that if Q is irrational and $d \geq 3$, then $\mathrm{vol}_{\mathbb{Z}}\, E_{a,b} = \infty$, whenever $a < b$. The study of the distribution of values of Q at integer points, often referred to as "quantitative Oppenheim conjecture" was the

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subject of several papers.
Let ρ be a continuous positive function on the sphere $\{v \in \mathbb{R}^d : ||v|| = 1\}$ and let $\Omega = \{v \in \mathbb{R}^d : ||v|| < \rho(v/||v||)\}$. We denote by $r\Omega$ the dilate of Ω by $r > 1$. In [DM93] S.G. Dani and G. Margulis obtained the following asymptotic exact lower bound under the same assumptions that Q is irrational and $d \geq 3$:

$$\liminf_{r\to\infty} \frac{\mathrm{vol}_{\mathbb{Z}}\,(E_{a,b} \cap r\Omega)}{\mathrm{vol}\,(E_{a,b} \cap r\Omega)} \geq 1. \tag{1.1}$$

Remark 1.1. It is not difficult to prove (see Lemma 3.8 in [EMM98]) that as $r \to \infty$,

$$\mathrm{vol}\,(E_{a,b} \cap r\Omega) \sim \lambda_{Q,\Omega}(b-a)r^{d-2},$$

where

$$\lambda_{Q,\Omega} \stackrel{\mathrm{def}}{=} \int_{L\cap\Omega} \frac{dA}{||\nabla Q||}, \tag{1.2}$$

L is the light cone $Q = 0$ and dA is the area element on L.

The situation with asymptotics and upper bounds is more subtle. It was proved in [EMM98] that if Q is an irrational indefinite quadratic form of signature (p, q), $p + q = d$, with $p \geq 3$ and $q \geq 1$, then for any $a < b$

$$\lim_{r\to\infty} \frac{\mathrm{vol}_{\mathbb{Z}}\,(E_{a,b} \cap r\Omega)}{\mathrm{vol}\,(E_{a,b} \cap r\Omega)} = 1 \tag{1.3}$$

or, equivalently, as $r \to \infty$,

$$\mathrm{vol}_{\mathbb{Z}}\,(E_{a,b} \cap r\Omega) \sim \lambda_{Q,\Omega}(b-a)r^{d-2}, \tag{1.4}$$

where $\lambda_{Q,\Omega}$ is as in (1.2).

If the signature of Q is $(2, 1)$ or $(2, 2)$ then no universal formula like (1.4) holds. In fact, it is proved in [EMM98] that if Ω is the unit ball and $q = 1$ or 2, then for every $\varepsilon > 0$ and every $a < b$ there exist an irrational quadratic form Q of signature $(2, q)$ and a constant $c > 0$ such that for an infinite sequence $r_j \to \infty$

$$\mathrm{vol}_{\mathbb{Z}}\,(E_{a,b} \cap r\Omega) > c r_j^q (\log r_j)^{1-\varepsilon}.$$

While the asymptotics as in (1.4) does not hold in the case of signatures $(2, 1)$ and $(2, 2)$, it is shown in [EMM98] that in this case there is an upper bound of the form $r^j \log r$. This upper bound is effective and it is uniform over compact sets in the space of quadratic forms. It is also shown in [EMM98] that there is an effective uniform upper bound of the form cr^{d-2} for the case $p \geq 3$, $q \geq 1$.

The examples in [EMM98] for the case of signatures $(2, 1)$ and $(2, 2)$ are obtained by considerations of irrational forms which are very well approximated by split rational forms. More precisely, a quadratic form Q is called *extremely well approximable by split rational forms* (EWAS) if for any $N > 0$ there exists a split integral form Q' and $2 \leq k \in \mathbb{R}$ such that

$$||kQ - Q'|| \leq \frac{1}{k^N},$$

where $||\cdot||$ denotes a norm on the linear space of quadratic forms. It is shown in [EMM98] that if Q is an indefinite quadratic form of signature $(2,2)$ which is not EWAS then for any interval (a,b), as $r\to\infty$,

$$\widetilde{N}_{Q,\Omega}(a,b,r)\sim\lambda_{Q,\Omega}(b-a)r^2, \tag{1.5}$$

where $\lambda_{Q,\Omega}$ is the same as in (1.2) and $\widetilde{N}_{Q,\Omega}(a,b,r)$ counts all the integral points in $E_{a,b}\cap r\Omega$ not contained in rational subspaces isotropic with respect to Q. It should be noted that

(i) an irrational quadratic form of signature $(2,2)$ may have at most four rational isotropic subspaces
(ii) if $0\notin(a,b)$, then $\widetilde{N}_{Q,\Omega}(a,b,r)=\mathrm{vol}_{\mathbb{Z}}\,(E_{a,b}\cap r\Omega)$.

The above mentioned results have analogs for inhomogeneous quadratic forms

$$Q_\xi[x]=Q[x+\xi],\quad \xi\in\mathbb{R}^d.$$

We define for $a,b\in\mathbb{R}$ with $a<b$ the shifted hyperbolic shell

$$E_{a,b,\xi}\overset{\text{def}}{=}\{x\in\mathbb{R}^d:\,a<Q_\xi[x]<b\}.$$

We say that Q_ξ is *rational* if there exists $t>0$ such that the coefficients of tQ and the coordinates of $t\xi$ are integers; otherwise Q_ξ is *irrational*. Then, under the assumptions that Q_ξ is irrational and $d\geq 3$, we have that (see [MM10])

$$\liminf_{r\to\infty}\frac{\mathrm{vol}_{\mathbb{Z}}\,(E_{a,b,\xi}\cap r\Omega)}{\mathrm{vol}\,(E_{a,b,\xi}\cap r\omega)}\geq 1. \tag{1.6}$$

The proof of (1.6) is similar to the proof of (1.1).

Let (p,q) be the signature of Q. If $p\geq 3$, $q\geq 1$, and Q_ξ is irrational then (see [MM10])

$$\lim_{r\to\infty}\frac{\mathrm{vol}_{\mathbb{Z}}\,(E_{a,b,\xi}\cap r\Omega)}{\mathrm{vol}\,(E_{a,b,\xi}\cap r\Omega)}=1 \tag{1.7}$$

or, equivalently, as $r\to\infty$,

$$\mathrm{vol}_{\mathbb{Z}}\,(E_{a,b,\xi}\cap r\Omega)\sim\lambda_{Q,\Omega}(b-a)r^{d-2}. \tag{1.8}$$

The proof of (1.7) is similar to the proof of (1.3). The paper [MM10] also contains an analog of (1.5) for inhomogeneous forms in the case of signature $(2,2)$. One should also mention related results of Marklof [Mark02, Mark03].

Remark 1.2. The proofs of the above mentioned results use such notions as a minimal invariant set (in the case of the Oppenheim conjecture) and an ergodic invariant measure. These notions do not have in general effective analogs. Because of that it is very difficult to get "good" estimates for the size of the smallest nontrivial integral solution of the inequality $|Q[m]|<\varepsilon$ and "good" error terms in the quantitative Oppenheim conjecture by applying dynamical and ergodic methods.

1.1. **Diophantine inequalities.** In the next sections we shall develop effective analogs of the results above which are needed to show that for *irrational* Q , $|Q[m]| < \varepsilon$ admits a nontrivial integral solution m with size measured in terms of ε^{-1}. For rational Q recall the following results for integer valued quadratic forms from reduction theory and the geometry of numbers. Let $A[m]$ denote an integer valued indefinite quadratic form on a d-dimensional lattice Λ in R^d. Meyer (1884), [Mey84], showed that such a form has a non-trivial zero on Λ if $d \geq 5$. Moreover, Birch and Davenport (1958), [BD58b], showed that

Theorem 1.3. *If $A[m]$ admits a nontrivial zero on Λ then there exist an $m \in \Lambda$ with Euclidean norm*

$$0 < \|m\| \leq \gamma_d (2 \operatorname{Tr} A^2)^{(d-1)/2} (\det \Lambda)^2, \tag{1.9}$$

where γ_d denotes Hermite's constant.

We shall use these results together with effective errors bounds for lattice point approximations to describe solutions of the Diophantine inequality $|Q[m]| < 1$ on $\mathbb{Z}^d$ for *all* indefinite forms.

Let Q denote as well the symmetric matrix in $\mathrm{GL}(d, \mathbb{R})$ associated with the form $Q[x] \stackrel{\text{def}}{=} \langle x, Q\, x\rangle$, where $\langle \cdot, \cdot \rangle$ is the standard Euclidean scalar product on $\mathbb{R}^d$. Let Q_+ denote the unique positive symmetric matrix such that $Q_+^2 = Q^2$ and let $Q_+[x] = \langle x, Q_+\, x\rangle$ denote the associated positive form with eigenvalues being the absolute eigenvalues of Q. Let q, resp. q_0 denote the largest resp. smallest of these absolute eigenvalues of Q and assume $q_0 \geq 1$.

Theorem 1.4. *For all indefinite forms nondegenerate forms of dimension $d \geq 5$ there exists an constant $c_{d,\eta}$, depending on d and $\eta > 0$ only, such that there is nontrivial integral solution of $|Q[m]| < 1$ with*

$$0 < Q_+[m] \leq c_{d,\eta} |\det Q|\, q^{\frac{4\,d}{d-4}+\eta}, \tag{1.10}$$

for arbitrary small $\eta > 0$.

Note that for a compact set of forms Q, such that $1 \leq q_0 \leq q \leq C(d)$, for some constant $C(d)$ depending on d only, we obtain nontrivial integral solutions of $|Q[m]| < \varepsilon$ (via rescaling $|(Q/\varepsilon)[m]| < 1$) with norms bounded by $\|m\| \leq c_{d,\eta} \varepsilon^{-d+1-\frac{2d}{d-4}-\eta}$ for $\eta > 0$ abitrary small. Embedding $\mathbb{Z}^5 \subset \mathbb{Z}^d$ for dimensions $d > 5$ (i.e. choosing $d-5$ coordinates to be zero), solutions of $|Q[m_0]| < 1$ for the case of 5 dimensions provide solutions of this Diophantine inequality for $d > 5$. Thus the case of $d = 5$ is of particular importance. Here Theorem 1.4 yields a nontrivial integral solution of $|Q[m]| < \varepsilon$ of size $0 < |m| \leq c_{d,\eta}\, \varepsilon^{-12-\eta}$. For the special case of *diagonal* indefinite forms $Q[x] = \sum_{j=1}^5 q_j x_j^2$ with $\min|q_j| \geq 1$ Birch and Davenport (1958), [BD58a], obtained a sharper bound. They showed for arbitrary small $\eta > 0$ that there exists an $m \in \mathbb{Z}^5 \setminus 0$ with $|Q[m]| < 1$ and $Q_+[m] \ll_{d,\eta} (\det Q_+)^{1+\eta}$. This implies (as above) for that compact set of forms Q that there exists an integral vector m with $\|m\| \leq c_{d,\eta}\, \varepsilon^{-2+\eta}$ and $0 < |Q[m]| < \varepsilon$.

For *irrational indefinite* quadratic forms we may quantify the density of values $Q[m]$ $m \in r\Omega \cap \mathbb{Z}^d$ as follows. Let $c_0 > 0$ denote a positive constant. Consider the set

$$V(r) \stackrel{\text{def}}{=} \big\{ Q[m] : \; m \in C_{r/c_0} \cap Z^d \big\} \cap [-c_0 \, r^2, c_0 \, r^2]$$

of values of $Q[x]$ lying in the interval $[-c_0 \, r^2, c_0 \, r^2]$, for $x \in C_{r/c_0}$. We define the maximal gap between successive values as

$$d(r) \stackrel{\text{def}}{=} \sup_{u \in V(r)} \inf \big\{ v - u \; : \; v > u, v \in V(r) \big\}. \tag{1.11}$$

As a consequence of our quantitative bounds in Theorems 2.1, 2.5 below we obtain

Corollary 1.5. *Let Q denote a non-degenerate d-dimensional indefinite form, which satisfies a Diophantine condition of type (κ, A) in* (1.15) *below for $\kappa > 0$ sufficiently small. Then for a fixed sufficiently small constant $c_0 > 0$, there exists a constant $c_{\kappa,A,Q,d}$ (explicitly depending on κ, A, Q and d) such that the maximal gap is bounded from above by*

$$d(r) \leq c_{d,\kappa,A,Q} \, r^{-\nu_0}, \tag{1.12}$$

for sufficiently large r, where $\nu_0 \stackrel{\text{def}}{=} (1-\kappa)(1-(4+\delta)/d)$ and $0 < \delta < 1/10$ is defined in Theorem 2.1.

For *positive definite* quadratic forms, Davenport and Lewis [DL72] conjectured in 1972, that the distance between successive values v_n of the quadratic form $Q[x]$ on $\mathbb{Z}^d$ converges to zero as $n \to \infty$, provided that the dimension d is at least five and Q is irrational. This conjecture was proved in Götze [Göt04].
The conjecture follows by the techniques of the present paper as well which provides error bounds for the lattice point counting problem *for the indefinite case as well as the positive definite case*.

The proof is similar as in the case of positive forms solved in [Göt04]: For any $\varepsilon > 0$ and any interval $[b, b+\varepsilon]$, we find at least two lattice points in the shell $E_{b,b+\varepsilon}$ (and the box of size $r = 2\sqrt{b}$), by Corollary 2.3, provided that b is larger than a threshold $b(\varepsilon)$. Here $b(\varepsilon)$ and consequently the distance between successive values (as a function of b) depends on the rate of convergence of the Diophantine characteristic $\rho(r)$, (in the bound of Corollary 2.3), towards zero.

1.2. **Discussion of Effective Bounds and Outline of Proofs.** In order to prove an effective result like Theorem 1.4 we need an explicit bound for the error, say $\delta(r\Omega \cap E_{a,b})$, of approximating the number of integral points $m \in E_{a,b}$ in a bounded domain $r\,\Omega$ by the volume in (1.2). Since the description of these error bounds is more involved for general domains Ω, we simplify the problem and first replace the weights 1 of integral points $m \in r\,\Omega$ by suitable smoothly changing weights $w(m)$, which tend to zero as m/r tends to infinity.

Smooth weights in $\mathbb{R}^d$. Using the weights $w(x) \stackrel{\text{def}}{=} \exp\{-Q_+[x]/r^2\}$ our techniques

yield effective bounds for the approximation of a weighted count of lattice points m with $Q[m] \in [a,b]$ by a corresponding integral with an error

$$\delta(r, E_{a,b}) \stackrel{\text{def}}{=} |\sum_{m \in E_{a,b}} w(m) - \int_{E_{a,b}} w(x)dx|. \tag{1.13}$$

The following bounds for $\delta(r, E_{a,b})$ are *identical* for the case of positive and indefinite d-dimensional forms Q, provided that $d \geq 5$. Using Vinogradov's notation $A \ll_B C$, (meaning that $A < c_B C$ with a constant $c_B > 0$ depending on B), we have

$$\delta(r, E_{a,b}) \quad \ll_{Q,d} \quad r^{d-2}(b-a)\rho_{Q,b-a}(r) + r^{d/2}\frac{b-a}{r}, \tag{1.14}$$

provided that $b - a \leq r$. (If $r < b - a \ll r^2$ the second term in the bound has to be replaced by $r^{d/2} \log r$). The function $\rho_{Q,b-a}(r)$ tends to zero for r tending to infinity if Q is irrational. Moreover, assume that Q is Diophantine in the sense that there exist constants $(\kappa,\ A)$, $\kappa \in (0,1)$ and $A > 0$, such that for every integer matrix M and integer $q \neq 0$ we have

$$\inf_{t \in [1,2]} \|Mq^{-1} - tQ\| > Aq^{-1-\kappa}. \tag{1.15}$$

Then we conclude that $\rho_Q(r, b-a) \ll_{Q,d,A} r^{-\nu_1}(b-a)^{1-\nu_2}$, where $\nu_1, \nu_2 \in (0,1)$ depend on d, κ and A (see Corollary 2.7).

Constant weights in $r\Omega$. Consider now regions Ω introduced in (1.1) above.

Introduce a w-smoothed indicator function, say $I^w_{a,b}(Q[x])$, for $E_{a,b}$ and an ε-smoothed indicator function, say $I^{\varepsilon}_{r\Omega}(x)$ for $r\Omega$. These smoothing procedures interpolate weights from 1 to 0 in the w- resp. the ε-boundaries of $[a,b]$ resp. $r\Omega$. Provided that we can control both errors by estimating the volumes of ε-boundaries, (compare Lemma 3.3, Corollary 3.2, (3.19)), we then have to estimate

$$\sum_{m \in \mathbb{Z}^d} w(m) I^w_{a,b}(Q[m]) I^{\varepsilon}_{r\Omega}(m) - \int_{\mathbb{R}^d} w(x) I^w_{a,b}(Q[x]) I^{\varepsilon}_{r\Omega}(x) dx \stackrel{\text{def}}{=} V_r - W_r, \quad \text{say}, \tag{1.16}$$

in order to bound the lattice point counting error $\delta(r\Omega \cap E_{a,b})$. Rewrite the weights in V_r (which are 1 in the interior of $r\Omega \cap E_{a,b}$) as product of the three bounded weights $w(m)$,(introduced above), $w_1(m) \stackrel{\text{def}}{=} I^w_{a,b}(Q[m])$ and $w_2(m) \stackrel{\text{def}}{=} \exp\{Q_+[m]/r^2\} I^{\varepsilon}_{r\Omega}(m)$. Using inverse Fourier transforms we may express these weights as

$$w_1(m) = \int \exp\{-itQ[m]\}\mu_{[a,b],w}(t)dt, \quad w_2(m) = \int_{\mathbb{R}^d} \exp\{-i\langle v, m\rangle\}\mu_{\Omega,\varepsilon}(v)dv,$$

with respect to some finite (signed) measures $\mu_{[a,b],w}$ and $\mu_{\Omega,\varepsilon}$.

Combining the factors $\exp\{-itQ[m]\}$, $\exp\{-\langle v, m\rangle\}$ in (1.16) and $w(m)$ into terms of the generalized theta series

$$\theta_v(t) \stackrel{\text{def}}{=} \sum_{m \in \mathbb{Z}^d} \exp\{-i\langle v, m\rangle - itQ[m] - Q_+[m]/r^2\}$$

one arrives at an expression for V_r by the following integral (in t and v) over $\theta_v(t)$:

$$V_r = \int_{\mathbb{R}^d} dv \mu_{\Omega,\varepsilon}(v) \int dt \mu_{[a,b],w}(t) \theta_v(t). \tag{1.17}$$

The approximating integral W_r to this sum V_r can be rewritten in exactly the same way by means of a theta integral, say $\Theta_v(t)$, replacing the theta sum $\theta_v(t)$. Thus, in order to estimate the error $|V_r - W_r|$, the integral over t and v of $|\theta_v(t) - \Theta_v(t)||\mu_{[a,b],w}(t)\mu_{\Omega,\varepsilon}(v)|$ has to be estimated.

For $|t| \le r^{-1}$ and $\|x\| \ll r$ the functions $x \mapsto \exp\{itQ[x]\}$ are sufficiently smooth, so that the sum $\theta_v(t)$ is well approximable by the first term of its Poisson series, that is the corresponding integral $\Theta_v(t)$, (see Lemma 4.3). The error of this approximation, after integration over v, yields the second error term in (1.14), which does not depend on the Diophantine properties of Q. The remaining error term we have to consider (for an appropriate choice of $T \sim w^{-1}$) is

$$I = \int_{T > |t| > r^{-1}} dt \int_{\mathbb{R}^d} dv |\theta_v(t)\, \mu_{[a,b],w}(t)\, \mu_{\Omega,\varepsilon}(v)|, \tag{1.18}$$

which is estimated as follows for $\kappa \in (0,1)$

$$I \le \|\mu_{\Omega,\varepsilon}\|_1 \sup_{T > |t| > r^{-1}} |\theta_v(t)|^{\kappa} \int_{T \ge |t| > r^{-1}} dt \sup_v |\theta_v(t)|^{1-\kappa} |\mu_{[a,b],w}(t)|, \tag{1.19}$$

where $\|\mu_{\Omega,\varepsilon}\|_1 = \int |\mu_{\Omega,\varepsilon}| dv$ depends on the shape of the region Ω and grows with the reciprocal of the ε-smoothing of the region. The second factor in the bound of I in (1.19) encodes the Diophantine behavior of Q as described above. The third factor is crucial. For some sufficiently small κ (depending on d) this average over t is of order r^{d-2}, provided that $d > 4$, see Lemma 7.1. The resulting bound, (when choosing $w \sim T^{-1}$ as order of the smoothing parameter for the region $E_{a,b}$), is an error bound of the form (compare Theorem 2.1)

$$\delta(r\Omega \cap E_{a,b}) \ll_{d,Q} \varepsilon(b-a)r^{d-2} + \|\mu_{\Omega,\varepsilon}\|_1 \rho(r, b-a) r^{d-2} + r^{d/2}\frac{b-a}{r}, \tag{1.20}$$

which has to be optimized in the smoothing size ε restricted to $\varepsilon \gg_d r^{-1}(\log r)^2$. The first term is due to the intersection of $E_{a,b}$ with the boundary of $r\Omega$. In the case of *positive definite* forms Q this intersection is empty for sufficiently large r, say $r = 2\sqrt{b}$, when $E_{a,b} \subset r/2\Omega$. Hence we may drop this term and fix ε, say at $1/16$. (Compare Theorem 2.3).

The Diophantine factor $\rho(r, b-a)$. In order to describe the second term in (1.20), we show that (uniformly in v (see Lemma 4.4)

$$|\theta_v(t)|^2 \le r^d \sum_{v \in \Lambda_t} \exp\{-\|v\|^2\} \overset{\text{def}}{=} I_{t,r}, \tag{1.21}$$

where $\Lambda_t \in SL(2d,\mathbb{R})/SL(2d,\mathbb{Z})$ is a family of $2d$-dimensional lattices generated by orbits of one-parameter-subgroups of $SL(2d,\mathbb{R})$ indexed by t and r.
The average of $I_{t,r}$ over t is derived in Lemmas 6.11,7.1 and 7.2, using an involved recursion in the size of r and building on a method developed in [EMM98] about upper estimates of averages of certain functions on the space of lattices along translates of orbits of compact subgroups. At this point the current approach is fundamentally different to the approach of previous effective bounds for $\delta(r\Omega \cap E_{a,b})$ by Bentkus and Götze [BG99], see also [BG97], valid for $d \ge 9$ and positive as well as indefinite forms. The reduction to $I_{t,r}$ and $\rho(r, b-a)$ follows the approach used in Götze, [Göt04], where $I_{t,r}$ has been estimated for $d \ge 5$ by methods from Geometry of Numbers, which *essentially required positive definite* forms. A variant of that method has been applied to *split* indefinite forms in a PhD thesis by Elsner [Els09].

In order to study the existence of solutions to the Diophantine inequalities in Theorem 1.4 with explicit dependence on the eigenvalues of Q, we have restricted ourselves to consider special regions Ω which are transformed cubes. This choice is motivated by the fact that in this case the first factor $\|\mu_{\Omega,\varepsilon}\|_1$ in the error bound (1.19) has an almost minimal growth $(\log \varepsilon^{-1})^d$ in ε^{-1}, see Lemma 4.1.

The paper is organized as follows. In the following Section 2 we describe the explicit technical estimates of lattice point remainders. In Section 3 we transfer the problem to Fourier space starting with a smoothing step and rewriting it in terms of integrals over d-dimensional theta sums. Section 4 provides a reformulation of the problem via upper bounds in terms of integrals over the absolute value of other theta sums with symplectic structure on $\mathbb{R}^{2d}$. These in turn are estimated using basic arguments from Geometry of Numbers, (Section 5) and in Section 6 based on crucial estimates for averages of functions on the space of lattices. Finally, in Section 7 all these results are combined to conclude the results of Section 2.

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2. Effective Estimates

We consider the quadratic form

$$Q[x] \overset{\text{def}}{=} \langle Qx, x\rangle, \quad \text{for} \quad x \in \mathbb{R}^d,$$

where $\langle \cdot, \cdot \rangle$ denotes the standard Euclidean scalar product and $Q : \mathbb{R}^d \to \mathbb{R}^d$ denotes a symmetric linear operator in $\mathrm{GL}(d, \mathbb{R})$ with eigenvalues, say, $q_1, \ldots, q_d$. Write

$$q_0 \stackrel{\text{def}}{=} \min_{1 \le j \le d} |q_j|, \qquad q \stackrel{\text{def}}{=} \max_{1 \le j \le d} |q_j|. \tag{2.1}$$

In what follows we shall always assume that the form is *non-degenerate*, that is, that $q_0 > 0$.

Using the matrix Q_+ defined in the introduction let $L_Q \stackrel{\text{def}}{=} Q_+^{1/2} \in \mathrm{GL}(d, \mathbb{R})$ denote the positive definite symmetric matrix with $L_Q^2 = Q_+$.

For $r > 0$ we consider as a special class of regions Ω the 'rescaled' boxes $r\Omega \stackrel{\text{def}}{=} C_r \stackrel{\text{def}}{=} \{L_Q x \in \mathbb{R}^d \;:\; |x|_\infty \le r\}$, where $|\cdot|_\infty$ denotes the maximum norm on $\mathbb{R}^d$, and

$$H_r \stackrel{\text{def}}{=} H_r^{a,b} \stackrel{\text{def}}{=} E_{a,b} \cap C_r. \tag{2.2}$$

We want to investigate the approximation of the lattice volume of H_r by the Lebesgue volume. Hence we estimate the following *relative* lattice point remainder of hyperbolic (or elliptic) shells in C_r for r large. Define

$$\Delta(r) \stackrel{\text{def}}{=} \left| \frac{\mathrm{vol}_{\mathbb{Z}} H_r - \mathrm{vol}\, H_r}{\mathrm{vol}\, H_r} \right|. \tag{2.3}$$

In order to describe the explicit bounds we need to introduce some more notations. Let $\beta > 2/d$ such that $0 < 1/2 - \beta < 1/2 - 2/d$ for $d > 4$. For a lattice $\Lambda \subset \mathbb{R}^{2d}$, with $\dim \Lambda = 2\,d$ and $1 \le l \le d$ we define its α_l-characteristic by

$$\alpha_l(\Lambda) \stackrel{\text{def}}{=} \sup\Big\{ \big|\det(\Lambda')\big|^{-1} : \Lambda' \subset \Lambda, \quad l\text{-dimensional sublattice of } \Lambda \Big\}. \tag{2.4}$$

Here, $\det(\Lambda') = \det(A^T A)^{1/2}$, where $\Lambda' = A\,\mathbb{Z}^d$ and A denotes a $(2d) \times d$-matrix. Introduce

$$\gamma_{[T_-,T],\beta}(r) \stackrel{\text{def}}{=} \sup\, \Big\{ \Big(r^{-d} \alpha_d(\Lambda_t) \Big)^{1/2-\beta} \;:\; T_- \le |t| \le T \Big\}, \tag{2.5}$$

where $\Lambda_t = d_r u_t \Lambda_Q$ denotes a $2d$-dimensional lattice obtained by the diagonal action of $d_r, u_t \in \mathrm{SL}(2, \mathbb{R})$ on $(\mathbb{R}^2)^d$ defined in (5.19) and Λ_Q denotes a fixed $2d$-dimensional lattice depending on Q introduced in (5.18).

With these notations we state a result providing quantitative bounds for the difference between the volume and the lattice point volume of a hyperbolic shell.

Theorem 2.1. *Let Q denote a non-degenerate d-dimensional indefinite form, $d \ge 5$, with $q_0 \ge 1$. Choose $\beta = 2/d + \delta/d$ for some arbitrary small $\delta \in (0, 1/10)$ and let $\varsigma \stackrel{\text{def}}{=} d(1/2 - \beta) = d/2 - 2 - \delta$. Then there exist constants $c_0 > 0$and $a(d)$ depending on d only such that, for any $r > q \ge q_0 \ge 1$, $b > a$ and $|a| + |b| < c_0 r^2$, we have with*

$(b-a)^* \stackrel{\mathrm{def}}{=} \min(b-a,1)$ and for any $r^{-1}(\log r)^2 \ll_d \varepsilon < 1/9$, $0 < w < (b-a)^*/8$

$$\begin{aligned}
\Delta_r &\stackrel{\mathrm{def}}{=} \Big|\frac{\mathrm{vol}_{\mathbb{Z}} H(r)}{\mathrm{vol}\, H(r)} - 1\Big| \\
&\le a(d)\Big\{\varepsilon + \frac{w}{b-a} + (\log \varepsilon^{-1})^d \Big(A_Q \frac{\rho^*_{Q,b-a,w}(r)}{b-a} + r^{-d/2}\xi(r,b-a)\Big)\Big\}, \text{ where} \\
\rho^*_{Q,b-a,w}(r) &\stackrel{\mathrm{def}}{=} \inf\Big\{\log(\frac{(b-a)^*}{w})\gamma_{[T_-,T_+],\beta}(r) + c_Q (b-a)^* T_-^{\varsigma} \,:\, T_- \in [1/r,1]\Big\} \\
\xi(r,b-a) &\stackrel{\mathrm{def}}{=} \frac{r^2 \log(1+\frac{b-a}{r})}{b-a}, \quad T_+ \asymp_d \varepsilon^{-1}\Big(\log(T_- q^{-1/2})^{\varsigma}(b-a)^*\Big)^2,
\end{aligned}$$

$A_Q \stackrel{\mathrm{def}}{=} |\det Q|^{1/4-\beta/2} q$ and $c_Q \stackrel{\mathrm{def}}{=} |\det Q|^{1/4-\beta/2}$ as well. Here we denote by $A \asymp_d B$ quantities of equivalent size (up to constants depending on d only), i.e. $A \ll_d B \ll_d A$.

Remark 2.2. The bound in Theorem 2.1 holds for inhomogeneous quadratic forms $Q[x+\xi]$ uniformly in $|\xi|_\infty \le 1$ as well.

Our techniques apply, using the $\gamma_{[T_-,T_+],\beta}$-characteristic of irrationality used above, for positive forms as well. Simplifications arise from the fact that only the w-smoothing of $E_{a,b}$ has to be calibrated with other parameters of the error bound. Here we shall choose a special region $r\Omega = C_r$ in such a way that the lattice points in $E_{a,b}$ are contained in C_r and will be counted with weights 1 using the ε-smoothing of the indicator of C_r, in (1.16) with $\varepsilon = 1/16$.

Corollary 2.3. *Let Q denote a non-degenerate d-dimensional* **positive** *definite form with $d \ge 5$, and $q_0 \ge 1$. For any $b > q \ge q_0 \ge 1$ and $r = 2b^{1/2}$ we have*

$$|\mathrm{vol}_{\mathbb{Z}} H_r - \mathrm{vol}\, H_r| \ll_d r^{d-2}\rho_Q(r) + |\det Q|^{-1/2} r^{d/2}\log r, \tag{2.6}$$

where

$$\rho_Q(r) \stackrel{\mathrm{def}}{=} \inf\Big\{\log(T_-^{-1}q)\gamma_{[T_-,T_+],\beta}(r) + c_Q T_-^{\varsigma} \,:\, T_- \in [r^{-1},1]\Big\}, \tag{2.7}$$

where $T_+ = B_Q T_-^{-\varsigma} h(T_-)$, $B_Q = |\det Q|^{-(1/2-\beta)} q^{-3/2}$ and $h(T_-) \stackrel{\mathrm{def}}{=} \Big(\log(T_- q^{-1/2})^{\varsigma} q^{1/2}\Big)^2$. Furthermore, $\lim_{r\to\infty}\rho(r) = 0$, as r (and b) tend to infinity, provided that Q is irrational. This bound is related to the bound obtained in [Göt04].

In the following we shall simplify the rather implicit bound of Theorem 2.1 for various choices of the interval length $b-a > 0$ and (smoothing) parameters ε, w and T which will be optimized depending on r.

Remark 2.4. Note that in Theorem 2.1 we have

$$\xi(r,b-a) \ll r, \quad \text{if} \quad b-a \le r, \quad \xi(r,b-a) \ll \frac{r^2}{b-a}\log(r), \quad \text{if} \quad b-a > r. \tag{2.8}$$

Corollary 2.5. *With the assumptions of Theorem 2.1 we have for* $b-a \le 1$

$$\Delta_r \ll_d A_Q \, \frac{\rho_Q(r, b-a)}{b-a} + r^{-d/2}\xi(r, b-a),$$

with $T_+ \asymp_d h(T_-)T_-^{-\varsigma}$ *and*

$$\rho_{Q,b-a}(r) \stackrel{\text{def}}{=} \inf \Big\{ |\log T_-|^d \Big(\gamma_{[T_-,T_+],\beta}(r) \log \frac{(b-a)^*}{T_-^{\varsigma}} + c_Q \, (b-a)^* T_-^{\varsigma} \Big) \; : \; T_- \in [r^{-1} + r^{-1/\varsigma}, 1] \Big\}. \tag{2.9}$$

For any fixed $T > 1 > T_-$ and irrational Q it is shown in Lemma 5.11 that

$$\lim_{r\to\infty} \gamma_{[T_-,T],\beta}(r) = 0, \tag{2.10}$$

with a speed depending on the Diophantine properties of Q. This implies for fixed $b-a > 0$ that

$$\lim_{r\to\infty} \rho_{Q,b-a}(r) = 0. \tag{2.11}$$

and hence $\lim\limits_{r\to\infty} \Delta_r = 0$.
In order to get an effective bound we consider the following class of Diophantine Q.

Definition 2.6. We call Q Diophantine of type (κ,A) if there exists a number $0 < \kappa < 1$ and a constant $0 < A < \infty$ such that for any $m \in \mathbb{Z}$ there exists an $M \in M(d, \mathbb{Z})$ with

$$\inf_{t\in[1,2]} \| M - mtQ)\| \ge A\,|m|^{-\kappa}. \tag{2.12}$$

Corollary 2.7. *Consider quadratic forms* $Q[x]$ *with matrix* Q *which is Diophantine of type* (κ,A). *Choosing* β *such that* $\beta d = 2 + \delta$ *for some sufficiently small* $0 < \eta < 1/10$, *we have with* $\mu \stackrel{\text{def}}{=} 1/2 - \beta = 1/2 - (2+\delta)/d$ *and* $(b-a)^* = \min(b-a, 1)$

$$\rho_{Q,b-a}(r) \ll_{d,\kappa,A,Q} (\log r)^{2d} r^{-\nu_1} \left((b-a)^*\right)^{1-\nu_2}, \tag{2.13}$$

where $\nu_1 = 2(1-\kappa)/(d+1+\kappa)$, *and* $\nu_2 = 1/(d+1+\kappa)/\mu$, *provided that* $r \ge 1$ *is sufficiently large and* $\kappa > 0$ *is sufficiently small (dending on* d*). Thus we conclude for sufficiently small and fixed* δ *and 'good' Diophantine types* (A, κ) *(with sufficiently small fixed* $\kappa > 0$*)*

$$\Delta(r) \ll_{d,A,\kappa,Q} (\log r)^{2d} r^{-\nu_1} ((b-a)^*)^{-\nu_2} + r^{-d/2}\xi(r, b-a) \tag{2.14}$$

$\xi(r, b-a) \stackrel{\text{def}}{=} \dfrac{r^2 \log(1 + \frac{b-a}{r})}{b-a}$, *where the implied constant in* (2.14) *can be explicitly determined.*

3. Fourier Analysis

3.1. **Smoothing.** The first step in the proof of Theorem 2.1 is the rewriting of lattice point counting errors $\bigl|\mathrm{vol}_{\mathbb{Z}} H_r - \mathrm{vol}\, H_r\bigr|$ in terms of integrals over appropriate smooth functions.

To this end we introduce a continuous approximation of the indicator function of $H_r = E_{a,b} \cap C_r$.

Denote by k a positive measure with compact support satisfying the assumptions $k([-1,1]) = 1$ and $|\widehat{k}(t)| \le \exp\{-c|t|^{1/2}\}$ for all $t \in \mathbb{R}$ and a positive constant c, where $\widehat{k}(t)$ is the characteristic function of the measure k (see [BR86]).

Let $1/9 > \varepsilon > 0$ and $w > 0$. Let k_ε and k_w denote the rescaled measures $k_\varepsilon(A) \stackrel{\mathrm{def}}{=} k(\varepsilon^{-1}A)$ and $k_w(A) \stackrel{\mathrm{def}}{=} k(w^{-1}A)$ for $A \in \mathcal{B}^1$. We write $\varepsilon_d \stackrel{\mathrm{def}}{=} (\varepsilon, \dots, \varepsilon) \in \mathbb{R}^d$, $\tau \stackrel{\mathrm{def}}{=} (\varepsilon_d, w) \in \mathbb{R}^{d+1}$ and let $k_\tau \stackrel{\mathrm{def}}{=} k_{\varepsilon,d} \times k_w$. Here $k_{\varepsilon,d} \stackrel{\mathrm{def}}{=} k_\varepsilon \times \cdots \times k_\varepsilon$, d times, denotes the product measure on $\mathbb{R}^d \times \mathbb{R}$. For $x = (x_1, \dots, x_{d+1}) \in \mathbb{R}^{d+1}$ let $B_\tau(x) \stackrel{\mathrm{def}}{=} \{y \in \mathbb{R}^{d+1} : |(y-x)_j| \le \varepsilon, j = 1, \dots, d, |(y-x)_{d+1}| \le w\}$ denote a box centered at x. We need the following lemma (which follows from [BR86]).

Lemma 3.1. *Let μ and ν be (positive) measures on $\mathbb{R}^{d+1}$ and let ε, w denote two positive constants as above. Then for every bounded real-valued, Borel-measurable function f on $\mathbb{R}^{d+1}$ we have*

$$\Bigl|\int f d(\mu-\nu)\Bigr| \le \max_{\pm} \Bigl|\int f_\tau^{\pm} d(\mu-\nu) * k_\tau\Bigr| + \int (f_{2\tau}^{+} - f_{2\tau}^{-}) d\nu, \tag{3.1}$$

where

$$f_\tau^{+}(x) \stackrel{\mathrm{def}}{=} \sup\{f(y) : y \in B_\tau(x)\} \quad \text{and} \quad f_\tau^{-}(x) \stackrel{\mathrm{def}}{=} (-f)_\tau^{+}(x). \tag{3.2}$$

Specializing (3.1) to $f = I_{C_r} I_{[a,b]}$, $a < b$, and $0 < 2w < b - a$, $\varepsilon < 1/9$ we obtain

Corollary 3.2. *Introduce $I_{1\pm\varepsilon}^{\varepsilon} \stackrel{\mathrm{def}}{=} I_{[-(1\pm\varepsilon), 1\pm\varepsilon]^d} * k_{\varepsilon,d}$ and $g_w \stackrel{\mathrm{def}}{=} I_{[a,b]}^{w} \stackrel{\mathrm{def}}{=} I_{[a,b]} * k_w$. Then we have*

$$\begin{aligned} &|\mathrm{vol}_{\mathbb{Z}^d}(C_r \cap E_{a,b}) - \mathrm{vol}_{\mathbb{R}^d}(C_r \cap E_{a,b})| \\ &\le \mathit{sup}^* \Bigl| \sum_{m \in \mathbb{Z}^d} I_{1\pm\varepsilon}^{\varepsilon}(L_Q\, m/r) I_{[a',b']}^{w}(Q[m]) - \int_{\mathbb{R}^d} I_{1\pm\varepsilon}^{\varepsilon}(L_Q\, x/r) I_{[a',b']}^{w}(Q[x])\, dx \Bigr| + R_{\varepsilon,w,r}, \end{aligned} \tag{3.3}$$

where $\sup^*$ denotes the supremum over the choices $\pm\varepsilon$ and $a' \in [a-w, a+w]$, $b' \in [b-w, b+w]$. Furthermore we introduced

$$\begin{aligned} R_{\varepsilon,w,r} \stackrel{\mathrm{def}}{=} &\int_{\mathbb{R}^d} I\Bigl(L_Q\, x/r \in [1-2\varepsilon, 1+2\varepsilon],\ Q[x] \in [a-2w, b+2w]\Bigr)\, dx + \\ &\int_{\mathbb{R}^d} I\Bigl(L_Q\, x/r \in [0, 1+2\varepsilon],\ Q[x] \in [a-2w, a+2w] \cup [b-2w, b+2w]\Bigr)\, dx. \end{aligned} \tag{3.4}$$

Proof. Let Λ^d denote the counting measure on $\mathbb{Z}^d$ and λ^d denote the Lebesgue measure on $\mathbb{R}^d$. Consider the map $T(x) \overset{\text{def}}{=} (T_1(x), T_2(x)) \overset{\text{def}}{=} (L_Q(x), Q[x])$ for $x \in \mathbb{R}^d$. Denote by Λ^d_T and λ^d_T the induced measures on $\mathbb{R}^{d+1}$ given by $\Lambda^d_T(A) \overset{\text{def}}{=} \Lambda^d(T^{-1}A)$ and $\lambda^d_T(A) \overset{\text{def}}{=} \lambda^d(T^{-1}A)$ for $A \in \mathcal{B}^{d+1}$. Using change of variables in the Lebesgue integral we obtain,

$$\begin{aligned}
&\mathrm{vol}_{\mathbb{Z}^d}(C_r \cap E_{a,b}) - \mathrm{vol}_{\mathbb{R}^d}(C_r \cap E_{a,b}) \\
&= \sum_{m \in \mathbb{Z}^d} I_{[-1,1]^d}(L_Q\, m/r) I_{[a,b]}(Q[m]) - \int_{\mathbb{R}^d} I_{[-1,1]^d}(L_Q\, x/r) I_{[a,b]}(Q[x])\, dx \\
&= \int_{\mathbb{R}^d} f(s,t)\, d(\Lambda^d_T - \lambda^d_T), \qquad (3.5)
\end{aligned}$$

where $f(s,t) \overset{\text{def}}{=} I_{[-1,1]^d}(s/r) I_{[a,b]}(t)$, $s \in \mathbb{R}^d$, $t \in \mathbb{R}$. Using Lemma 3.1 we obtain the estimate

$$\Big|\int_{\mathbb{R}^d} f(s,t)\, d(\Lambda^d_T - \lambda^d_T)\Big| \quad \le \quad \max_{\pm}\Big|\int_{\mathbb{R}^d} f^{\pm}(s,t)\, d(\Lambda^d_T - \lambda^d_T) * k_\tau\Big| + \Big|\int_{\mathbb{R}^d} \Delta_{2\tau} f(s,t)\, d\lambda^d_T\Big|,$$

$$\begin{aligned}
\text{where} \qquad f^+(s,t) &\overset{\text{def}}{=} I_{[-1-\varepsilon,1+\varepsilon]^d}(s) I_{[a-w,b+w]}(t), \\
f^-(s,t) &\overset{\text{def}}{=} I_{[-1+\varepsilon,1-\varepsilon]^d}(s) I_{[a+w,b-w]}(t), \\
\text{and} \qquad \Delta_{2\tau} f(s,t) &\overset{\text{def}}{=} I_{[0,1+2\varepsilon]^d}(s) I_{[a-2w,a+2w]\cup[b-2w,b+2w]}(t) + \\
&\qquad + I_{[0,1+2\varepsilon]^d \setminus [0,1-2\varepsilon]^d}(s) I_{[a-2w],b+2w]}(t). \qquad (3.6)
\end{aligned}$$

The assertion of Corollary 3.2 now follows from (3.5) and (3.6). □

Lemma 3.3. *The following bounds hold for any $\varepsilon \in (0, 1/9)$ and $|a| + |b| < c_0 r^2$, $r > q^{1/2}$ (with a constant $0 < c_0 = c_0(d) < 1/8$ depending on the dimension only)*

$$\begin{aligned}
R_{\varepsilon,w,r} &\ll_d |\det Q|^{-1/2}(\varepsilon(b-a) + w) r^{d-2}, \qquad (3.7) \\
\mathrm{vol}\, H_r &\gg_d |\det Q|^{-1/2}(b-a) r^{d-2}. \qquad (3.8)
\end{aligned}$$

For a related bound to (3.8) *see (*[BG99]*, p. 1023, Lemma 8.2 or* [EMM98]*, Lemma 3.8). Here we need an effective version in terms of Q.*

Proof. Assume that $\frac{|a|+|b|}{r^2} < c_0$, where c_0 is an absolute constant to be determined later. Let $J_1 \overset{\text{def}}{=} I_0 \times [a,b]$. For the bound (3.7) choose $I_0 \overset{\text{def}}{=} [1-2\varepsilon, 1+2\varepsilon] \overset{\text{def}}{=} I_{1,2\varepsilon}$ with $0 < \varepsilon < \frac{1}{9}$. whereas for the bound (3.8) choose $I_0 \overset{\text{def}}{=} [0,1] \times [a,b]$.

Consider the region $V_1 \overset{\text{def}}{=} \mathrm{vol}\{x \in \mathbb{R}^d : (||L_Q x||_\infty / r, Q[x]) \in J_1\}$. Let U denote a rotation in $\mathbb{R}^d$ such that UQU^{-1} and hence UL_QU^{-1} are diagonal. Changing variables via $y = U^{-1}L_Q x/r$ in $\mathbb{R}^d$ with $y \in \mathbb{R}^p \times \mathbb{R}^q, d = p + q, y = (y_1, y_2)$ and $S_0[y] =$

$\|y_1\|^2 - \|y_2\|^2$ we may rewrite the region V_1 as

$$\begin{aligned} V_1 &= r^d\,|\det Q|^{-1/2}\overline{V}_1, \quad \text{where} \\ \overline{V}_1 &\overset{\text{def}}{=} \operatorname{vol}\{y \in \mathbb{R}^d : (||Uy||_\infty, r^2 S_0[y]) \in J_1\} \\ &= \{y \in \mathbb{R}^d \,:\, \|U\,y\|_\infty \in I_0,\ S_0[y] \in [a/r^2, b/r^2]\}. \end{aligned} \tag{3.9}$$

Write $y = (y_1, y_2) = (r_1\eta_1, r_2\eta_2)$ with $\eta_1 \in S^p$, $\eta_2 \in S^q$, $||\eta_1|| = ||\eta_2|| = 1$ and $r_1, r_2 \geq 0$. By Fubini's theorem, we have

$$\overline{V}_1 = \int_0^\infty \int_0^\infty r_1^{p-1} r_2^{q-1} I(r_1^2 - r_2^2 \in [a/r^2, b/r^2])\varphi(r_1, r_2)\, dr_1 dr_2,$$

where

$$\varphi(r_1, r_2) = \int_{S^p \times S^q} I(||U(r_1\eta_1, r_2\eta_2)||_\infty \in I_0)\, d\eta_1 d\eta_2.$$

Note that, for $|v| \leq c_0$, $v \overset{\text{def}}{=} r_1^2 - r_2^2$, $u \overset{\text{def}}{=} r_1$ we have $r_1^2 + r_2^2 = 2u^2 + v$. In order to estimate $\overline{V}_1$ rewrite $r_2 = \sqrt{u^2 - v}$ in terms of v. We have by change of variables

$$\overline{V}_1 \asymp_d \int_{a/r^2}^{b/r^2} \left(\int_0^{c(d)} I(u^2 \geq v)\, u^{p-1} \varphi(u, \sqrt{u^2 - v})(u^2 - v)^{(q-2)/2}\, du \right) dv, \tag{3.10}$$

Proof of (3.8)**.** Since here $I_0 = [0, 1]$, $|v| \leq c_0$ and $u \leq c(d)$ we conclude that

$$\overline{V}_1 \gg_d (b - a)r^{-2},$$

which proves (3.8).

Proof of (3.7)**.** Let $(e_m, m = 1, \ldots, d)$ denote the standard orthonormal basis in $\mathbb{R}^d$ and let $(f_m, m = 1, \ldots, d)$, $f_k = U^{-1}e_m$, be the transformed basis. Since $I(\|U\,y\|_\infty \in I_{1,2\varepsilon}) \leq \sum_{m=1}^d I(|\langle y, f_m\rangle| \in I_0)$ we get

$$\varphi(r_1, r_2) \leq \sum_{m=1}^d \varphi_m(r_1, r_2), \quad \text{where} \quad \varphi_m(r_1, r_2) \overset{\text{def}}{=} \int_{S^p \times S^q} I\Big(|\langle(r_1\eta_1, r_2\eta_2), f_m\rangle| \in I_{1,2\varepsilon}\Big)\, d\eta_1\, d\eta_2.$$

Recall $|v| \leq c_0, v = r_1^2 - r_2^2, u = r_1$ and $r_2 = \sqrt{u^2 - v}$. Then the norm inequalities between $||\cdot||_\infty$ and $||\cdot||$ imply $d(1 + 2\varepsilon)^2 \geq r_1^2 + r_2^2 = 2u^2 + v \geq (1 - 2\varepsilon)^2$. Thus

$$\varphi(u, \sqrt{u^2 - v}) = 0 \quad \text{if} \quad 0 \leq u \leq \frac{1 - 2\varepsilon}{\sqrt{2}} \quad \text{or} \quad u > c(d) \overset{\text{def}}{=} \frac{(1 + 2\varepsilon)d}{\sqrt{2}}.$$

Since $(1-2\varepsilon)/\sqrt{2} > 1/2$ we have as in (3.10)

$$\begin{aligned}\overline{V}_1 &\ll \int\limits_{a/r^2}^{b/r^2} \Bigg(\int\limits_{1/2}^{c(d)} u^{p-1} \varphi(u, \sqrt{u^2-v}) (u^2-v)^{(q-2)/2}\, du \Bigg) dv \\ &\le \sum_{m=1}^{d} \int\limits_{a/r^2}^{b/r^2} \Bigg(\int\limits_{1/2}^{c(d)} u^{p-1} \varphi_m(u, \sqrt{u^2-v}) (u^2-v)^{(q-2)/2}\, du \Bigg) dv. \qquad (3.11)\end{aligned}$$

Since $\sqrt{u^2-v} \le \sqrt{u^2+c_0} \ll_d 1$, we see that

$$\int\limits_{1/2}^{c(d)} u^{p-1} (u^2-v)^{(q-2)/2} \varphi_m(u, \sqrt{u^2-v})\, du \ll_d R_m \stackrel{\text{def}}{=} \int\limits_{1/2}^{c(d)} u^{p-1} \varphi_m(u, \sqrt{u^2-v})\, du.$$

We claim that

$$V_1 \ll_d |\det Q|^{-1/2} \varepsilon (b-a) r^{d-2} \qquad (3.12)$$

holds. In view of (3.9), (3.10) and (3.11), the estimates $R_m \ll_d \varepsilon$ for all $m = 1, \dots, d$ together will prove the bound (3.12).

Thus let $F_m(u) \stackrel{\text{def}}{=} |\langle (u\eta_1, \sqrt{u^2-v}\,\eta_2), f_m \rangle|$ for fixed $|v| \le c_0$ and (η_1, η_2). If

$$\Big| \frac{\partial}{\partial u} F_m(u) \Big| \ge c_1 > 0 \qquad (3.13)$$

for all $\frac{1}{2} \le u \le c(d)$, then

$$\int\limits_{1/2}^{c(d)} I(F_m(u) \in I_{1,2\varepsilon})\, du \ll \frac{\delta}{c_1}$$

and hence $R_m \ll_d \varepsilon$ for all $m = 1, \dots, d$. Note that $\frac{\partial}{\partial u} F_m(u) = \pm \langle (\eta_1, \frac{u}{\sqrt{u^2-v}} \eta_2), f_m \rangle$. Furthermore, the function $\zeta(v) \stackrel{\text{def}}{=} \frac{u}{\sqrt{u^2-v}} - \frac{\sqrt{u^2-v}}{u}$ satisfies the relations $\zeta(0) = 0$ and $|\zeta(v)| \le c_0 \max_{0 \le |v| \le c_0} |\zeta'(v)|$. Since $\zeta'(v) = \frac{2u^2-v}{2u(u^2-v)^{3/2}}$, we get

$$\max_{0 \le |v| \le c_0} |\zeta'(v)| < \frac{2u^2+c_0}{2u(u^2-c_0)^{3/2}}.$$

Since $\frac{1}{2} \le u \le c(d)$, we have

$$\Big| \frac{\partial}{\partial u} F_m(u) \Big| \ge \Big| \frac{F_m(u)}{u} \Big| - |\zeta(v)| \ge \frac{1-\varepsilon}{u} \Big(1 - c_0 \frac{u + c_0/2}{(u^2-c_0)^{3/2}} \Big) \ge \frac{3}{8u} \ge \frac{3}{8c(d)},$$

for $0 < c_0 < \frac{1}{9}$ sufficiently small. Therefore (3.13) holds and the assertion (3.12) is proved. This yields the desired bound for first integral in $R_{\varepsilon,w,r}$ (see (3.4)). To estimate

the second integral in $R_{\varepsilon,w,r}$ it remains to prove that

$$V_2 \ll_d |\det Q|^{-1/2} w r^{d-2}, \tag{3.14}$$

where $V_2 \overset{\text{def}}{=} \mathrm{vol}\{x \in \mathbb{R}^d : (||L_Q x||_\infty / r, Q[x]) \in J_2\}$ and $J_2 \overset{\text{def}}{=} [0,1] \times ([a-2w, a+2w] \cup [b-2w, b+2w])$. Using the previous arguments with some simplifications, we obtain the estimate (3.14) (see [BG99] as well). The statement (3.7) of the lemma then follows from (3.12) and (3.14). □

Introduce the functions

$$\psi(x) \overset{\text{def}}{=} \varphi\Big(\frac{1}{d}||x||^2\Big) \exp\{||x||^2\}. \tag{3.15}$$

Here $\varphi = I_{[-2,2]} * k$ denotes a C^∞-function such that $\varphi(u) = 1$ if $|u| \le 1$ and $\varphi(u) = 0$ if $|u| \ge 3$.

In order to introduce exponential convergence factors, rewrite $I^{\varepsilon}_{1\pm\varepsilon}(L_Q x/r)$ as follows

$$I^{\varepsilon}_{1\pm\varepsilon}(L_Q x/r) = \chi_{\pm\varepsilon,r}(x) \exp\Big\{ -r^{-2} Q_+[x]\Big\}, \tag{3.16}$$

where

$$\chi_{\pm\varepsilon,r}(x) \overset{\text{def}}{=} I^{\varepsilon}_{1\pm\varepsilon}(L_Q x/r)\psi\big(\frac{1}{r}L_Q x\big).$$

Furthermore, in order to simplify notations write $g_w \overset{\text{def}}{=} I^w_{[a,b]}$ and for $w, \varepsilon > 0$

$$V^{\mathbb{Z}}_{w,\pm\varepsilon}(r;a,b) \overset{\text{def}}{=} \sum_{x \in \mathbb{Z}^d} \exp\Big[-r^{-2}\, Q_+[x]\Big]\, g_w\Big(Q[x]\Big)\, \chi_{\pm\varepsilon,r}(x) dx \tag{3.17}$$

and

$$V^{\mathbb{R}}_{w,\pm\varepsilon}(r;a,b) \overset{\text{def}}{=} \int_{\mathbb{R}^d} \exp\Big[-r^{-2}\, Q_+[x]\Big]\, g_w\Big(Q[x]\Big)\, \chi_{\pm\varepsilon,r}(x) dx \quad . \tag{3.18}$$

Using these notations, Corollary 3.2 and Lemma 3.3 yield the upper bound

$$\begin{aligned} |\mathrm{vol}_{\mathbb{Z}^d}(C_r \cap E_{a,b}) - \mathrm{vol}_{\mathbb{R}^d}(C_r \cap E_{a,b})| \;\le\; & \max_{\pm\varepsilon} |V^{\mathbb{Z}}_{w,\pm\varepsilon}(r;a,b) - V^{\mathbb{R}}_{w,\pm\varepsilon}(r;a,b)| \\ & + c(d)\, |\det Q|^{-1/2} (\varepsilon(b-a) + w) r^{d-2}. \end{aligned} \tag{3.19}$$

4. Estimates by Theta-series

We want to estimate the above errors by Fourier inversion and thus by integrals over theta functions.

For $v \in \mathbb{C}^d$ we introduce the following theta sum and theta integral

$$\theta_v(z) \quad \stackrel{\text{def}}{=} \quad \sum_{x \in \mathbb{Z}^d} \exp\left[-Q_{r,v}(t,x)\right], \tag{4.1}$$

$$\widetilde{\theta}_v(t) \quad \stackrel{\text{def}}{=} \quad \int_{R^d} \exp\left[-Q_{r,v}(t,x)\right] dx, \tag{4.2}$$

where $Q_{r,v}(t,x) \stackrel{\text{def}}{=} r^{-2} Q_+[x] - it \cdot Q[x] - i \cdot \langle x, \frac{v}{r} \rangle$.
Note that

$$|\widehat{g}_w(t)| \ll \min\{|b-a|, |t|^{-1}\} \exp\{-c|tw|^{1/2}\}. \tag{4.3}$$

Now we can rewrite the right hand side of (3.19) as follows

$$\left|V^{\mathbb{Z}}_{w,\varepsilon}(r;a,b) - V^{\mathbb{R}}_{w,\pm\varepsilon}(r;a,b)\right| = \Big| \int\limits_{-\infty}^{\infty} \widehat{g}_w(t) \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} \widehat{\chi}_{\pm\varepsilon,r}(v) \cdot \left\{\theta_v(t) - \widetilde{\theta}_v(t)\right\} dv \; dt \Big|.$$

Consider the segments $J_0 \stackrel{\text{def}}{=} [-\frac{1}{r}; \frac{1}{r}]$ and $J_1 \stackrel{\text{def}}{=} \mathbb{R} \setminus J_0$. We may split the integral on the right hand side as follows into terms

$$\left|V^{\mathbb{Z}}_{w,\pm\varepsilon}(r;a,b) - V^{\mathbb{R}}_{w,\pm\varepsilon}(r;a,b)\right| \ll_d I_{0,\pm} + I^*_{0,\pm} + I_{1,\pm}, \quad \text{say, where} \tag{4.4}$$

$$I_{0,\pm} \quad \stackrel{\text{def}}{=} \quad \Big| \int\limits_{J_0} \widehat{g}_w(t) \int_{\mathbb{R}^d} \widehat{\chi}_{\pm\varepsilon,r}(v) \cdot \left\{\theta_v(t) - \widetilde{\theta}_v(t)\right\} dv \; dt \Big|, \tag{4.5}$$

$$I^*_{0,\pm} \quad \stackrel{\text{def}}{=} \quad \Big| \int\limits_{J_1} \widehat{g}_w(t) \int_{\mathbb{R}^d} \widehat{\chi}_{\pm\varepsilon,r}(v) \cdot \widetilde{\theta}_v(t) dv \; dt \Big| \quad \text{and} \tag{4.6}$$

$$I_{1,\pm} \quad \stackrel{\text{def}}{=} \quad \Big| \int\limits_{J_1} \widehat{g}_w(t) \int_{\mathbb{R}^d} \widehat{\chi}_{\pm\varepsilon,r}(v) \cdot \theta_v(t) dv \; dt \Big|. \tag{4.7}$$

Before proceeding with the estimation of the terms $I_{0,\pm}$, $I_{1,\pm}$ we shall collect pointwise bounds of theta series and integrals in the following Lemmas using Poisson's Formula. Furthermore, we need bounds for integrals of $\widehat{\chi_{\pm\varepsilon,r}}$. Starting with the latter we have

Lemma 4.1. *The following estimates hold*

$$\int\limits_{\mathbb{R}^d} |\widehat{\chi}_{\pm\varepsilon,r}(v)| \, dv \quad \le \quad c(d)(\log\frac{1}{\varepsilon})^d \quad \textit{and} \tag{4.8}$$

$$\int\limits_{\|L_Q^{-1} v\|_\infty > R} |\widehat{\chi}_{\pm\varepsilon,r}(v)| \, dv \quad \le \quad c(d,k)\frac{1}{R^k}(\log\frac{1}{\varepsilon})^d, \tag{4.9}$$

for arbitrary large $k \in \mathbb{N}$ and for all R such that $R(\log R)^{-2} \ge c(d)k^2\varepsilon^{-1}$ with a sufficiently large constant $c(d)$.

Proof. We shall prove this lemma for the function $\widehat{\chi}_{+\varepsilon,r}$. One can prove the assertion of the lemma for $\widehat{\chi}_{-\varepsilon,r}$ in the same way. Note that by definition (3.16)

$$\int\limits_{\mathbb{R}^d} |\widehat{\chi}_{\varepsilon,r}(v)|\,dv = \int\limits_{\mathbb{R}^d} |\widehat{I^{\varepsilon}_{1+\varepsilon}\psi}(v)|\,dv = \int\limits_{\mathbb{R}^d} \Big| \int\limits_{\mathbb{R}^d} \widehat{I^{\varepsilon}_{1+\varepsilon}}(v-x)\widehat{\psi}(x)\,dx \Big|\,dv. \tag{4.10}$$

Hence $\int\limits_{\mathbb{R}^d} |\widehat{\chi}_{\varepsilon,r}(v)|\,dv \le \|\widehat{I}^{\varepsilon}_{1+\varepsilon}\|_1 \|\widehat{\psi}\|_1$, where $\|f\|_p$ denotes the p-norm, $p \in [1,\infty]$, of f with respect to Lebesgue-measure. Since $|D_1^{\alpha_1}\dots D_d^{\alpha_d}\psi(x)| \le c(d,\alpha_1,\dots,\alpha_d)$, we easily conclude that

$$|\widehat{\psi}_r(x)| \le c(d,k)(1+||x||^2)^{-k}, \quad x \in \mathbb{R}^d \quad \text{and} \tag{4.11}$$

$$\|\widehat{\psi}\|_1 \le c(d). \tag{4.12}$$

Thus we have

$$\int\limits_{\mathbb{R}^d} |\widehat{I}^{\varepsilon}_{1+\varepsilon}(v)|\,dv = \int\limits_{\mathbb{R}^d} |\widehat{I}_{[-(1+\varepsilon),1+\varepsilon]^d}(v)||\widehat{k}_{\varepsilon,d}(v)|\,dv \le c^d \Big(\int\limits_{\mathbb{R}} \Big| \frac{\sin\frac{1}{2}(1+\varepsilon)v}{v} \Big| |\widehat{k}(\varepsilon v)|\,dv \Big)^d \le$$

$$\le c^d \Big(1 + \int\limits_{|v|\ge 1} \frac{1}{|t|} |\widehat{k}(\varepsilon v)|\,dv \Big)^d \le c^d \Big(1 + \int\limits_{|v|\ge \varepsilon^{1/2}} \frac{1}{|v|} e^{-c|v|^{1/2}}\,dv \Big)^d \le c^d (\log\frac{1}{\varepsilon})^d. \tag{4.13}$$

We obtain the estimate (4.8) from (4.12) and (4.13). Furthermore,

$$\begin{aligned} &\int\limits_{||L_Q^{-1}v||_\infty > R} |\widehat{\chi}_{\varepsilon,r}(v)|\,dv \\ \le\; &\int\limits_{|v||_\infty > R,\, ||x||_\infty \le \frac{1}{2}R} |\widehat{I}^{\varepsilon}_{1+\varepsilon}(v-x)||\widehat{\psi}(x)|\,dx\,dv + \int\limits_{|v||_\infty > R,\, ||x||_\infty > \frac{1}{2}R} |\widehat{I}^{\varepsilon}_{1+\varepsilon}(v-x)||\widehat{\psi}(x)|\,dx\,dv \\ \le\; &\int\limits_{|v||_\infty > \frac{1}{2}R} |\widehat{I}^{\varepsilon}_{1+\varepsilon}(v)|\,dv\,||\widehat{\psi}||_1 + ||\widehat{I}^{\varepsilon}_{1+\varepsilon}||_1 \int\limits_{||x||_\infty > \frac{1}{2}R} |\widehat{\psi}(x)|\,dx. \end{aligned} \tag{4.14}$$

By (4.11), we have

$$\int\limits_{||x||_\infty > \frac{1}{2}R} |\widehat{\psi}(x)|\,dx \le c(k)\frac{1}{R^k}. \tag{4.15}$$

Repeating the argument which we employed in the proof of (4.13), we arrive at the following bound, (for $\frac{R}{\log^2 R} \geq \frac{ck^2}{\varepsilon}$ and $k \geq cd$ with sufficiently large $c > 0$),

$$\begin{aligned} \int\limits_{|v||_\infty > \frac{1}{2}R} |\widehat{I}^{\varepsilon}_{1+\varepsilon}(v)|\, dv \;&\leq\; c(d)\Big(\int\limits_{\mathbb{R}} \Big|\frac{\sin\frac{1}{2}(1+\varepsilon)v}{v}\Big| |\widehat{k}(\varepsilon v)|\, dv\Big)^{d-1} \int\limits_{|v|\geq R/(2d)} \frac{1}{|v|}|\widehat{k}(\varepsilon v)|\, dv \\ &\leq\; c(d)(\log\frac{1}{\varepsilon})^d e^{-c(d)(\varepsilon R)^{1/2}} \leq c(d)(\log\frac{1}{\varepsilon})^d \frac{1}{R^k}. \end{aligned} \tag{4.16}$$

The assertion (4.9) of the lemma follows directly from (4.14) and (4.12), (4.13), (4.15), and (4.16). $\square$

Concerning estimates of $\theta_v - \widetilde{\theta}_v$ we have

Lemma 4.2. *(Poisson's formula) For a symmetric, $d \times d$ complex matrix Ω, whose imaginary part is positive definite the following holds:*

$$\sum_{m\in\mathbb{Z}^d} \exp\Big\{\pi i\Omega[m] + 2\pi i\langle m, v\rangle\Big\} = \Big(\det\Big(\frac{\Omega}{i}\Big)\Big)^{-\frac{1}{2}} \sum_{n\in\mathbb{Z}^d} \exp\Big\{-\pi i\Omega^{-1}[n+v]\Big\}. \tag{4.17}$$

The term with $n = 0$ of the series over $n \in \mathbb{Z}^d$ is given by

$$\int_{\mathbb{R}^d} \exp\{\pi i\Omega[x] + 2\pi i\langle x, v\rangle\}dx = \left(\det\left(\frac{\Omega}{i}\right)\right)^{-\frac{1}{2}} \exp\left\{-\pi i\cdot\Omega^{-1}[v]\right\}. \tag{4.18}$$

Here $s \in \mathbb{R}^d$ and $\Omega^{-1}\big[x\big]$ denotes the quadratic form $\langle\Omega^{-1}x, x\rangle$, defined by the inverse operator $\Omega^{-1} : \mathbb{C}^d \to \mathbb{C}^d$ (which exists since Ω is an element of Siegel's generalized upper half plane).

Proof. See [Mum83], p. 195 (5.6) and Lemma 5.8. $\square$

Lemma 4.3. *Let $\theta_v(z)$ and $\widetilde{\theta}_v(z)$ denote the theta sum and the theta integral introduced in (4.1) and (4.2) respectively, that is*

$$\theta_v(z) \;\stackrel{\text{def}}{=}\; \sum_{x\in\mathbb{Z}^d} \exp\left[-Q_{r,v}(z,x)\right] \quad \textit{and} \tag{4.19}$$

$$\widetilde{\theta}_v(t) \;\stackrel{\text{def}}{=}\; \int_{R^d} \exp\left[-Q_{r,v}(z,x)\right] dx, \tag{4.20}$$

where $Q_{r,v}(z,x) \stackrel{\text{def}}{=} r^{-2}Q_+[x] - itQ[x] - i\cdot\langle x, \frac{v}{r}\rangle$.

Let $t \in \mathbb{R}$, $|t| < \frac{1}{r}$ and $r \geq 1$, then the following bound holds

$$\begin{aligned} &\big|\big(\theta_v - \widetilde{\theta}_v\big)(t)\big| \\ \ll_d\;& |r^{-2}+it|^{-\frac{d}{2}}\Big(\exp\big\{-\operatorname{Re}\big((r^{-2}+it)^{-1}\big)\big\} + |\det Q|^{-1/2}\, I_{(\pi r/q^{1/2},\infty)}(|Q_+^{-1/2}v|)\Big). \end{aligned}$$

Proof. Using Lemma 4.2 with $\Omega \overset{\mathrm{def}}{=} \frac{i}{\pi}\tilde{Q}_t$, where $\tilde{Q}_t = r^{-2}Q_+ + itQ$ is a selfadjoint operator, we get by (4.17) and (4.18) that

$$\big(\theta_v - \widetilde{\theta}_v\big)(t) \;=\; \det\Big(\pi^{-1}\tilde{Q}_t\Big)^{-\frac{1}{2}} \sum_{n\in\mathbb{Z}^d\setminus\{0\}} \exp\Big[-\tilde{Q}_t^{-1}\big[\pi n - \frac{\tilde{v}}{2r}\big]\Big]. \tag{4.21}$$

Note that $S \overset{\mathrm{def}}{=} Q_+^{-1}Q$ is a reflection (with eigenvalues ± 1) and Q, Q_+, S commute. Thus we obtain $\tilde{Q}_t = Q_+(r^{-2} + itS)$. Diagonalizing these operators, we get $\det\Big(\pi^{-1}\tilde{Q}_t\Big) = \det(\pi Q_+)\cdot|z|^d$, with $z \overset{\mathrm{def}}{=} r^{-2} + it$ and hence

$$\Big|\det\Big(\pi^{-1}\tilde{Q}_t\Big)\Big|^{-1/2} \le |\det(\pi Q)|^{-1/2}|z|^{-d/2}. \tag{4.22}$$

Similarly write

$$\mathrm{Re}(\tilde{Q}_t^{-1}) = Q_+^{-1}\,\mathrm{Re}\Big((r^{-2}I_d + itS)^{-1}\Big).$$

Hence, we get an inequality for positive symmetric matrices $\mathrm{Re}(z^{-1})\,\mathrm{Re}(\tilde{Q}_t^{-1}) \ge Q_+^{-1}$ which implies

$$\begin{aligned}\Big|\exp\Big\{-\pi^2\tilde{Q}_t^{-1}\big[n - \frac{v}{2\pi r}\big]\Big\}\Big| \;&=\; \exp\Big\{-\mathrm{Re}\Big(\tilde{Q}_t^{-1}\big[\pi n - \frac{v}{2r}\big]\Big)\Big\}\\ &\le\; \exp\Big\{-\mathrm{Re}(z^{-1})Q_+^{-1}[\pi n - \frac{v}{2r}]\Big\}.\end{aligned} \tag{4.23}$$

Using (4.21) and (4.23) we get

$$\big|\big(\theta_v - \widetilde{\theta}_v\big)(t)\big| \ll_d |\det Q|^{-1/2}|z|^{-d/2} \sum_{n\in\mathbb{Z}^d\setminus\{0\}} \exp\Big\{-\mathrm{Re}(z^{-1})Q_+^{-1}[\pi n - \frac{v}{2r}]\Big\}. \tag{4.24}$$

For all $\|Q_+^{-1/2}v\| \le \pi r/q^{1/2}$ and $n \in \mathbb{Z}^d\setminus\{0\}$ we obtain

$$\begin{aligned}Q_+^{-1}[\pi n - \frac{v}{2r}]^{1/2} \;&=\; \|Q_+^{-1/2}(\pi n - \frac{v}{2r})\| \ge \pi\|Q_+^{-1/2}n\| - \|Q_+^{-1/2}v\|/2r\\ &\ge\; \frac{\pi}{4}\|Q_+^{-1/2}n\| + \frac{\pi}{4q^{1/2}}.\end{aligned}$$

Thus (4.24) yields

$$\big|\big(\theta_v - \widetilde{\theta}_v\big)(t)\big| \;\ll_d\; K\cdot|z|^{-d/2}\exp\Big\{-\mathrm{Re}(z^{-1})\frac{\pi^2}{16q}\Big\}, \quad \text{where} \tag{4.25}$$

$$K \;\overset{\mathrm{def}}{=}\; |\det Q|^{-1/2}\sum_{n\in\mathbb{Z}^d}\exp\Big\{-\mathrm{Re}(\frac{\pi^2}{16z})Q_+^{-1}[n]\Big\}. \tag{4.26}$$

Let $I \overset{\mathrm{def}}{=} [-\frac{1}{2},\frac{1}{2}]^d$. Then $Q_+^{-1}[x] \le 1$, for $x \in I, q_0 \ge 1$, and

$$\int_I \exp\{-Q_+^{-1}[m+x]/2\}dx \ge \exp\{-Q_+^{-1}[m]/2 - d/2\}\int_I \exp\{-\langle Q_+^{-1}m, x\rangle\}dx,$$

where the integral on the right hand is larger than 1 by Jensen's inequality. Thus if $\mathrm{Re}(\frac{\pi^2}{8z}) \geq 1/2$ (which follows from $\mathrm{Re}(1/(r^{-2}+it) = r^2/(1+t^2r^4) \geq 4/\pi^2$ since $|t| < 1/r$) we obtain

$$K \ll_d \sum_{n\in\mathbb{Z}^d} |\det Q|^{-1/2} \int_{I+n} \exp\{-Q_+^{-1}[x]/2\}dx \ll_d 1, \tag{4.27}$$

which proves Lemma 4.3 for $\|Q_+^{-1/2}v\| \leq \pi r/q^{1/2}$. Otherwise, set $v = L\pi r + v'$, with $L \in \mathbb{Z}^d$, $\|Q_+^{-1/2}v'\| \leq \pi r/q^{1/2}$. By (4.21) we obviously have $\theta_v = \theta_{v'}$ and therefore

$$\big|\big(\theta_v - \widetilde{\theta}_v\big)(t)\big| \leq \big|\big(\theta_{v'} - \theta_{0,v'}\big)(t)\big| + \big|\big(\theta_{0,v'} - \theta_{0,v}\big)(t)\big|. \tag{4.28}$$

Since by (4.22) $|\theta_{0,v'}(t)| + |\theta_{0,v}(t)| \ll_d |\det Q|^{-1/2}|z|^{-d/2}$ the Lemma is proved in view of (4.25) and (4.27). $\square$

We now return to the error integral terms in (4.4). First we shall estimate the volume approximation term I_0.

Estimation of $I_{0,\pm}$: First we derive a bound for $I_{0,+}$. For $t \in J_0$ we obtain, using $z_t \stackrel{\text{def}}{=} r^{-2} + it$ and Lemma 4.3:

$$\begin{aligned}
\Theta_t &\stackrel{\text{def}}{=} \left|\int_{\mathbb{R}^d} \widehat{\chi}_{\varepsilon,r}(v)\cdot\big\{\theta_v(t) - \widetilde{\theta}_v(t)\big\}dv\right| \ll_d \Theta_{t,1} + \Theta_{t,2}, \quad \text{say, where} \qquad (4.29)\\
\Theta_{t,1} &\stackrel{\text{def}}{=} |z_t|^{-\frac{d}{2}} \exp\big\{-\mathrm{Re}\big(z_t^{-1}\big)\big\} \int_{\mathbb{R}^d} |\widehat{\chi}_{\varepsilon,r}(v)|\,dv, \quad \text{and}\\
\Theta_{t,2} &\stackrel{\text{def}}{=} |\det Q|^{-1/2}|z_t|^{-d/2} \int_{\mathbb{R}^d} |\widehat{\chi}_{\varepsilon,r}(v)|\, I_{(\pi r/q^{1/2},\infty)}(|Q_+^{-1/2}v|)dv.
\end{aligned}$$

Since $|z_t| = r^{-2}(1+r^4t^2)^{1/2}$ and $\mathrm{Re}\big(z_t^{-1}\big) = \frac{r^2}{1+r^4t^2}$, we may rewrite $\Theta_{t,1}$ via $s = (1+r^4t^2)^{-1}$ and the function $h(s) \stackrel{\text{def}}{=} s^{d/4}\exp\{-sr^2\}$. The maximal value of h on $[0,\infty)$ is attained at $s_0 = \dfrac{d}{4r^2}$ and it is bounded by $(r^{-2})^{d/4}$ up to a constant depending on d only. Hence we get by Lemma 4.1

$$\begin{aligned}
\sup_{t\in J_0} \Theta_{t,1} &\ll_d r^d \sup_{s\geq 0} h(s) \int_{\mathbb{R}^d} |\widehat{\chi}_{\varepsilon,r}(v)|\,dv \ll_d r^d \Big(r^2\Big)^{-\frac{d}{4}} \int_{\mathbb{R}^d} |\widehat{\chi}_{\varepsilon,r}(v)|\,dv\\
&\ll_d r^{d/2}(\log\varepsilon^{-1})^d, \qquad (4.30)
\end{aligned}$$

and similarly by Lemma 4.1 with $R = \pi r/q^{1/2}$ we have

$$\begin{aligned}
\sup_{t\in J_0} \Theta_{t,2} &\ll_{d,k} (\log\varepsilon^{-1})^d |\det Q|^{-1/2} (q^{1/2}/r)^k |z_t|^{-d/2} \ll_{d,k} (\log\varepsilon^{-1})^d q^{k/2} |\det Q|^{-1/2} r^{-k+d}\\
&\ll_{d,k} r^{d/2}|\det Q|^{-1/2}(\log\varepsilon^{-1})^d, \qquad (4.31)
\end{aligned}$$

provided that we choose $k = d$ and $r > q$. Thus we conclude by (4.29),(4.30) and (4.31)

$$\sup_{t \in J_0} \Theta_t \ll_d r^{d/2} (\log \varepsilon^{-1})^d. \tag{4.32}$$

Integrating this bound over $t \in J_0$ with weight $|\widehat{g}_w(t)|$, we get

$$|I_{0,+}| \leq \int_{-\frac{1}{r}}^{\frac{1}{r}} \Theta_t dt \ll_d r^{d/2} (\log \varepsilon^{-1})^d \int_{-\frac{1}{r}}^{\frac{1}{r}} \min\{(b-a), |t|^{-1}\} dt.$$

Thus we conclude

$$|I_{0,+}| \ll_d \log(1 + |b-a| r^{-1}) r^{d/2} (\log \varepsilon^{-1})^d. \tag{4.33}$$

It is easy to see that the bound for $I_{0,-}$ is of the same form.

Estimation of $I^*_{0,\pm}$: By (4.18) with $\Omega \stackrel{\text{def}}{=} \frac{i}{\pi} \tilde{Q}_t$, where $\tilde{Q}_t = r^{-2} Q_+ + itQ$, we conclude that $|\widetilde{\theta}_v(t)|$ in (4.2) may be estimated as in (4.23) and (4.22) (with $n = 0$)

$$|\widetilde{\theta}_v(t)| \ll_d |\det Q|^{-1/2} |z_t|^{-d/2} \exp\{-\operatorname{Re}(z_t^{-1}) Q_+^{-1}[\frac{v}{2r}]\}.$$

Thus we obtain using Lemma 4.1

$$I^*_{0,\pm} \ll_d (\log(\varepsilon^{-1})^d |\det Q|^{-1/2} \int_{\frac{1}{r}}^{\infty} |\widehat{g}_w(t)| |z_t|^{-d/2} dt.$$

Note that for $r^2 s > 1$ we have

$$\int_s^\infty |z_t|^{-d/2} \frac{dt}{t} \ll_d r^{d-2} (r^2 s)^{-d/2}. \tag{4.34}$$

Since $|\widehat{g}_w(t)| \leq \min\{|b-a|, |t|^{-1}\} \exp\{-c|wt|^{1/2}\}$ we obtain for the case $|b-a| < r$ using (4.34) for $s = 1/r$ and $s = r/|b-a|$, splitting the t-integral into the parts $\frac{1}{r} \leq |t| \leq r/|b-a|$ and $|t| \geq r/|b-a|$

$$I^*_{0,\pm} \ll_d (\log \varepsilon^{-1})^d \min\{|b-a|, r\} r^{-1} |\det Q|^{-1/2} r^{d/2} r^{-2}, \tag{4.35}$$

provided that $d > 2$. Inequality (4.35) holds as well in the case $|b-a| > r$ by similar arguments. Note that combining (4.33) and (4.35) we obtain in view of $|\det Q| \geq 1$

$$|I^*_{0,\pm}| + |I_{0,+}| \ll_d (\log \varepsilon^{-1})^d \log(1 + |b-a| r^{-1}) r^{d/2}. \tag{4.36}$$

By means of (3.3),(3.7), (4.4) and (4.36) we may now summarize the results obtained so far as follows. For any $1/9 > \varepsilon \gg_d r^{-1} (\log r)^2$ and $|a| + |b| \leq c_0 r^2$ we have

$$\Big|\operatorname{vol}_{\mathbb{Z}}(H_r) - \operatorname{vol}_{\mathbb{R}}(H_r)\Big| \ll_d \sup{}^*(I_{1,\pm}, I_{0,\pm}, I^*_{0,\pm}) + R_{\varepsilon,w,r} \tag{4.37}$$

$$\ll_d \quad \sup{}^* I_{1,\pm} + (\log \varepsilon^{-1})^d r^{d/2} \log(1 + |b-a| r^{-1}) + |\det Q|^{-1/2} (\varepsilon(b-a) + w) r^{d-2},$$

where $\sup^*$ denotes the sup over $\pm\varepsilon$ and $a' \in [a-w, a+w], b' \in [b-w, b+w]$ and

$$\begin{aligned} I_{1,\pm} &= \Big| \int_{J_1} \widehat{g}_w(t) \int_{\mathbb{R}^d} \widehat{\chi}_{\pm\varepsilon,r}(v)\theta_v(t) dv\, dt \Big| \\ &\ll_d (\log\varepsilon^{-1})^d \int_{|t|>\frac{1}{r}} |\widehat{g}_w(t)| \sup_v |\theta_v(t)|\, dt, \end{aligned} \tag{4.38}$$

in view of Lemma 4.1.

Estimation of $I_{1,\pm}$: We shall estimate $I_{1,+}$. The bounds for $I_{1,-}$ have the same form and are proved in exactly the same way. In the sequel we shall therefore I_1 instead of $I_{1,+}$. This is the crucial error part. At first we shall bound the theta series $\theta_v(t)$ uniformly in v by another theta series in dimension $2d$ in order to transform the problem to averages over functions on the space of lattices (subject to actions of $SL(2,\mathbb{R})$. We have

Lemma 4.4. *Let $\theta_v(t)$ denote the theta function in* (4.19) *depending on Q and $v \in \mathbb{C}^d$. For $r \geq 1$, $t \in \mathbb{R}$, the following bound holds*

$$\big|\theta_v(t)\big| \ll_d (\det Q_+)^{-1/4} r^{d/2}\, \psi(r,t)^{1/2}, \quad \textit{where} \tag{4.39}$$

$$\psi(r,t) \overset{\text{def}}{=} \sum_{m,n\in\mathbb{Z}^d} \exp\{-H_t(m,n)\}, \quad \textit{and} \tag{4.40}$$

$$H_t(m,n) \overset{\text{def}}{=} r^2 Q_+^{-1}[m - \frac{2}{\pi} t\, Qn] + r^{-2} Q_+[n], \tag{4.41}$$

and $H_t(m,n)$ is a positive quadratic form on $\mathbb{Z}^{2d}$.

Note that the right hand side of this inequality is independent of $v \in \mathbb{R}^d$.

Proof. For any $x, y \in \mathbb{R}^d$ the equalities

$$2\,(Q_+[x] + Q_+[y]) = Q_+[x+y] + Q_+[x-y], \tag{4.42}$$

$$\langle Q(x+y), x-y\rangle = Q[x] - Q[y] \tag{4.43}$$

hold. Rearranging $\theta_v(z)\,\overline{\theta_v(z)}$ and using (4.43), we would like to use $m+n$ and $m-n$ as new summation variables on a lattice. But both vectors have the same parity, i.e., $m+n \equiv m-n \mod 2$. Since they are dependent one has to consider the 2^d sublattices indexed by $\alpha = (\alpha_1, \dots, \alpha_d)$ with $\alpha_j = 0, 1$, for $1 \leq j \leq d$:

$$\mathbb{Z}^d_\alpha \overset{\text{def}}{=} \{m \in \mathbb{Z}^d : m \equiv \alpha \mod 2\},$$

where, for $m = (m_1, \dots, m_d)$, $m \equiv \alpha \mod 2$ means $m_j \equiv \alpha_j \mod 2$, $1 \leq j \leq d$. Thus writing

$$\theta_{v,\alpha}(t) \overset{\text{def}}{=} \sum_{m\in\mathbb{Z}^d_\alpha} \exp\left[-\frac{2}{r^2} Q_+[m] - it\cdot Q[m] + i\cdot\langle m, \frac{v}{r}\rangle\right],$$

we obtain $\theta_v(t) = \sum_\alpha \theta_{v,\alpha}(t)$ and hence by the Cauchy–Schwarz inequality

$$\big|\theta_v(t)\big|^2 \le \; 2^d \sum_\alpha \big|\theta_{v,\alpha}(t)\big|^2. \tag{4.44}$$

Using (4.43) and the absolute convergence of $\theta_\alpha(t)$, we may rewrite the quantity $\theta_{v,\alpha}(t)\,\overline{\theta_{v,\alpha}(t)}$ as follows:

$$\begin{aligned}
&\theta_{v,\alpha}(t)\,\overline{\theta_{v,\alpha}(t)} \\
&\quad = \sum_{m,n\in\mathbb{Z}^d_\alpha} \exp\Big[-\frac{1}{r^2}\big(Q_+[m]+Q_+[n]\big) - it\cdot\big(Q[m]-Q[n]\big) - i\cdot\langle m-n, \frac{v}{r}\rangle\Big] \\
&\quad = \sum_{m,n\in\mathbb{Z}^d_\alpha} \exp\Big[-\frac{2}{r^2}\big(Q_+[\overline{m}]+Q_+[\overline{n}]\big) - 2i\cdot\big\langle 2t\cdot Q\overline{m} + \frac{v}{r}, \overline{n}\big\rangle\Big]
\end{aligned} \tag{4.45}$$

where $\overline{m} = \dfrac{m+n}{2}$, $\overline{n} = \dfrac{m-n}{2}$. Note that the map $H : \bigcup_\alpha \mathbb{Z}^d_\alpha \times \mathbb{Z}^d_\alpha \to \mathbb{Z}^d \times \mathbb{Z}^d, (m,n) \longmapsto \Big(\dfrac{m+n}{2}, \dfrac{m-n}{2}\Big)$ is a bijection. Therefore we get by (4.44)

$$\begin{aligned}
&\big|\theta_v(t)\big|^2 \\
&\quad \ll_d \sum_{\alpha\in\{0,1\}^d} \sum_{\overline{m},\overline{n}\in\mathbb{Z}^d_\alpha} \exp\Big[-r^{-2}\big(Q_+[\overline{m}]+Q_+[\overline{n}]\big) - 2i\cdot\big\langle 2t\cdot Q\overline{m} + \frac{\tilde{v}}{r}, \overline{n}\big\rangle\Big] \\
&\quad = \sum_{\overline{m},\overline{n}\in\mathbb{Z}^d} \exp\Big[-\frac{2}{r^2}\big(Q_+[\overline{m}]+Q_+[\overline{n}]\big) - 2i\cdot\big\langle 2t\cdot Q\overline{m} + \frac{\tilde{v}}{r}, \overline{n}\big\rangle\Big].
\end{aligned} \tag{4.46}$$

In this double sum fix $\overline{n}$ and sum over $\overline{m} \in \mathbb{Z}^d$ first. Using Corollary 4.2 for $\Omega = (iQ_+r^{-2} + tQ)/\pi$, we get for $\delta \overset{\text{def}}{=} \big(\det\big(\frac{2}{\pi r^2}\cdot Q_+\big)\big)^{-\frac{1}{2}}$ by the symmetry of Q

$$\begin{aligned}
\theta_v(t,\overline{n}) \;&\overset{\text{def}}{=} \sum_{\overline{m}\in\mathbb{Z}^d} \exp\Big[-\frac{2}{r^2}\big(Q_+[\overline{m}]+Q_+[\overline{n}]\big) - 2i\cdot\big\langle 2t\cdot Q\overline{m} + \frac{\tilde{v}}{r}, \overline{n}\big\rangle\Big] \\
&= \delta \sum_{m\in\mathbb{Z}^d} \exp\Big[-\frac{r^2}{2}Q_+^{-1}[\pi m - 2t\,Q\overline{n}] - \frac{2}{r^2}Q_+[\overline{n}] - 2i\langle\frac{\tilde{v}}{r}, \overline{n}\rangle\Big].
\end{aligned}$$

Thus,

$$\big|\theta_v(t,\overline{n})\big| \le \delta \sum_{m\in\mathbb{Z}^d} \exp\Big\{-\frac{r^2}{2}Q_+^{-1}[\pi m - 2t\,Q\overline{n}] - \frac{2}{r^2}Q_+[\overline{n}]\Big\}. \tag{4.47}$$

Hence, we obtain by (4.46) and (4.47)

$$\big|\theta_v(t)\big|^2 \ll_d (\det Q_+)^{-1/2} r^d \sum_{m,\overline{n}\in\mathbb{Z}^d} \exp\Big\{-G_t[m,n]\Big\},$$

where $G_t[m,n] \overset{\text{def}}{=} \frac{r^2}{2} Q_+^{-1}[\pi m - 2t\, Q\overline{n}] + \frac{2}{r^2} Q_+[\overline{n}]$. Since $\pi^2/2 > 1$ we may bound $G_t[m,n]$ from below as follows:

$$G_t[m,n] \geq r^2 Q_+[m - \frac{2}{\pi} t Q\bar{n}] + r^{-2} Q_+[\bar{n}] = H_t[m,n],$$

which proves Lemma 4.4. □

Lemma 4.5. *Let Λ be a lattice in $\mathbb{R}^d$. Assume that $0 < \varepsilon \leq 1$. Then*

$$\exp\{-\varepsilon\} \#H \;\leq \sum_{v \in \Lambda} \exp\{-\varepsilon\, \|v\|^2\} \ll_d \varepsilon^{-d/2} \#H, \tag{4.48}$$

where $H \overset{\text{def}}{=} \{v \in \Lambda \,:\, \|v\|_\infty < 1\}$.

Proof. The lower bound for the sum is obvious by restricting summation to the set of elements in H. As for the upper bound introduce for $\mu = (\mu_1, \dots, \mu_d) \in \mathbb{Z}^d$ the sets

$$B_\mu \overset{\text{def}}{=} \Big[\mu_1 - \frac{1}{2}, \mu_1 + \frac{1}{2}\Big) \times \cdots \times \Big[\mu_d - \frac{1}{2}, \mu_d + \frac{1}{2}\Big)$$

such that $\mathbb{R}^d = \bigcup_\mu B_\mu$. For any fixed $w^* \in H_\mu \overset{\text{def}}{=} \{w \in \Lambda \cap B_\mu\}$ we have

$$w - w^* \in H, \qquad \text{for any } w \in H_\mu.$$

Hence we conclude for any $\mu \in \mathbb{Z}^d$

$$\#H_\mu \;\leq\; \#H.$$

Since $x \in B_\mu$ implies $\|x\|_\infty \geq \|\mu\|_\infty/2$, we thus obtain

$$\begin{aligned}
\sum_{v \in \Lambda} \exp\{-\varepsilon\, \|v\|^2\} \quad &\leq \quad \sum_{v \in \Lambda} \exp\{-\varepsilon\, \|v\|_\infty^2\} \\
&\leq \quad \#H_0 + \sum_{\mu \in \mathbb{Z}^d \setminus 0} \; \sum_{v \in \Lambda} \mathbf{I}\{\, v \in B_\mu\}\, \exp\{-\frac{\varepsilon}{4}\, \|\mu\|_\infty^2\} \\
&\leq \quad \#H \cdot \sum_{\mu \in \mathbb{Z}^d} \exp\{-\frac{\varepsilon}{4}\, \|\mu\|^2\} \\
&\ll_d \quad \varepsilon^{-d/2}\, \#H.
\end{aligned}$$

This concludes the proof of Lemma 4.5. □

5. Functions on the space of Lattices and Geometry of Numbers

Let $n \in \mathbb{N}^+$ and let $e_1, e_2, \dots, e_l$ be $l \leq n$ denote linearly independent vectors in $\mathbb{R}^n$. Let Λ be the *lattice of rank l* generated by all integer combinations of these vectors. Then the determinant of the lattice Λ, denoted by $\det(L)$, is given by $\det\big(\langle e_i, e_j\rangle, i, j = 1, \dots l\big)^{1/2}$. It does not depend on the choice of the basis of Λ, .

More generally, for applications in the next section, for every l, $1 \leq l \leq n$, we fix a quasinorm $|\cdot|_l$ on the exterior product $\wedge^l \mathbb{R}^n$. Let Δ be a lattice in $\mathbb{R}^n$. We say

that a subspace L of $\mathbb{R}^n$ is Δ-*rational* if $L \cap \Delta$ is a lattice in L. For any Δ-rational subspace L, we denote by $d_\Delta(L)$ or simply by $d(L)$ the quasinorm $|u_1 \wedge \ldots \wedge u_l|_l$ where $\{u_1, \ldots, u_l\}, l = \dim L$, is a basis of $L \cap \Delta$ over $\mathbb{Z}$. If $\{v_1, \ldots, v_l\}$ is another basis of $L \cap \Delta$, then $u_1 \wedge \ldots \wedge u_l = \pm v_1 \wedge \ldots \wedge v_l$. Hence $d(L)$ does not depend on the choice of basis in $L \cap \Delta$. For $L = \{0\}$ we write $d(K) \stackrel{\text{def}}{=} 1$. If the quasinorms $|\cdot|_i$ are norms on $\wedge^l \mathbb{R}^n$ induced from the Euclidean norm on $\mathbb{R}^n$, then $d(L)$ is equal to $\det(L)$, that is to the volume of $L/(L \cap \Delta)$. In particular, in this case the lattice Δ is unimodular iff $d_\Delta(\mathbb{R}^n) = 1$. Also in this case $d(L)d(M) \geq d(L \cap M)d(L + M)$ for any two Δ-rational subspaces L and M (see Lemma 5.6 in [EMM98]). But any two quasinorms on $\wedge^l \mathbb{R}^n$ are equivalent. This proves the following:

Lemma 5.1. *There is a constant $C \geq 1$ depending only on the quasinorm $|\cdot|_l$ and not depending on Δ such that*

$$C^2 d(L)d(M) \geq d(L \cap M)d(L + M) \tag{5.1}$$

for any two Δ-rational subspaces L and M.

Let us introduce the following notations for $0 \leq l \leq n$,

$$\alpha_l(\Delta) \stackrel{\text{def}}{=} \sup\{\frac{1}{d(L)} : L \text{ is a } \Delta - \text{rational subspace of dimension } l\}, \tag{5.2}$$

$$\alpha(\Delta) \stackrel{\text{def}}{=} \max_{0 \leq l \leq n} \alpha_l(\Delta). \tag{5.3}$$

This extends the earlier definition of $\alpha_i(\Delta)$, see (2.4), of Theorem 2.1 to the case of general seminorms on $\wedge^i \mathbb{R}^n$. In this section the functions α_l and α will be based on standard Euclidean norms though, that is, they use $d(L) = \det(L)$.

In the following we shall use some facts in the geometry of numbers (see Davenport (1958), [Dav58] or Cassels (1959) and classical reduction theory for lattices in $\mathbb{R}^n$, [Cas59]). They show that the number of points from Δ in the unit ball in $\mathbb{R}^n$ lies between two positive constants, (see Corollary 5.7 below).

Let $F :\to [0, \infty]$ denote a norm on $\mathbb{R}^n$. The successive minima $M_1 \leq \cdots \leq M_n$ of F with respect to a lattice Λ in $\mathbb{R}^n$ are defined as follows: Let $M_1 = \inf\{F(m) : m \neq 0, m \in \Lambda\}$ and define M_j as the infimum of $\lambda > 0$ such that the set $\{m \in \Lambda : F(m) < \lambda\}$ contains j linearly independent vectors. It is easy to see that these infima are attained, that is there exist linearly independent vectors $b_1, \ldots, b_n \in \Lambda$ such that $F(b_j) = M_j$, $j = 1, \ldots, n$.

Lemma 5.2. *Let $F_j(m)$, $j = 1, 2$, be some norms in $\mathbb{R}^n$ and $M_1 \leq \cdots \leq M_n$ and $N_1 \leq \cdots \leq N_n$ be the successive minima of F_1 with respect to the lattice Λ_1 and of F_2 with respect to the lattice Λ_2 respectively. Let $C > 0$. Assume that $M_k \gg_n C\, F_2(b_k)$, $k = 1, 2, \ldots, n$, for some linearly independent vectors $b_1, b_2, \ldots, b_n \in \Lambda_2$. Then*

$$M_k \gg_n C\, N_k, \quad k = 1, \ldots, n. \tag{5.4}$$

The proof of this lemma is elementary and therefore omitted.

For a lattice $\Lambda \subset \mathbb{R}^n$, $\dim \Lambda = n$ and $1 \le l \le n$ recall the definition of α_l-characteristics by

$$\alpha_l(\Lambda) \stackrel{\text{def}}{=} \sup\Big\{ \big|\det(\Lambda')\big|^{-1} : \Lambda' \subset \Lambda, \quad l\text{-dimensional sublattice of } \Lambda. \Big\} \tag{5.5}$$

Lemma 5.3. *Let $F(\cdot)$ be a norm in $\mathbb{R}^n$ such that $F(\cdot) \asymp_n \|\cdot\|$. Let $M_1 \le \dots \le M_n$ be the successive minima of F with respect to a lattice $\Lambda \subset \mathbb{R}^n$ of rank n. Then*

$$\alpha_l(\Lambda) \asymp_n (M_1 \cdot M_2 \cdots M_l)^{-1}, \quad l = 1, \dots, n. \tag{5.6}$$

Suppose that $1 \le j \le n$ and $M_j \le \mu < M_{j+1}$, for some $\mu > 0$ and $M_{n+1} = \infty$. Then

$$\#\{v \in \Lambda \, : \, F(v) \le \mu\} \asymp_n \mu^j (M_1 \cdot M_2 \cdots M_j)^{-1}. \tag{5.7}$$

For the proof of (5.6) in Lemma 5.3 we shall use the following result about reduced lattice bases formulated in Proposition (p. 517) and Remark (p. 518) in A.K. Lenstra, H.W. Lenstra and Lovász (1982), [LLL82].

Lemma 5.4. *Let $M_1 \le \dots \le M_n$ be the successive minima of the standard Euclidean norm with respect to a lattice $\Lambda \subset \mathbb{R}^n$. Then*

$$M_l \asymp_d \|e_l\|, \quad l = 1, \dots, n. \tag{5.8}$$

Moreover,

$$\det(\Lambda) \asymp_n \prod_{l=1}^{n} \|e_l\|. \tag{5.9}$$

Proof of Lemma 5.3.
Proof of (5.6):. According to Lemma 5.2, we can replace the Euclidean norm $\|\cdot\|$ by the norm $F(\cdot)$, in the formulation of Lemma 5.4. Let $\Lambda' \subset \Lambda$ be an arbitrary l-dimensional sublattice of Λ and $N_1 \le \dots \le N_l$ be the successive minima of the norm $F(\cdot)$ with respect to Λ'. It is clear that $M_j \le N_j$, $j = 1, 2, \dots, l$. If $M_j = F(b_j)$ for some linearly independent vectors $b_1, b_2, \dots, b_l \in \Lambda$ and

$$\Lambda' = \Big\{ \sum_{j=1}^{l} n_j b_j : n_j \in \mathbb{Z}, j = 1, 2, \dots, l \Big\},$$

then $M_j = N_j$, $j = 1, 2, \dots, l$. It remains to apply Lemma 5.4.
Proof of (5.7). Let a_l denote the elements in Λ corresponding to the successive minima M_l, $l = 1, \dots, n$. For any $v \in \Lambda$ with $F(v) \le M_j \le \mu$ we have by definition

$$v = m_1 a_1 + \ldots + m_j a_j$$

with some $m_j \in \mathbb{Z}$. Since for $|m_l| \le j^{-1} \mu F(a_l)^{-1}$, $l = 1, \dots, j$ we have $F(v) \le \mu$ we conclude that

$$N(\mu) \stackrel{\text{def}}{=} \#\{v \in \Lambda \, : \, F(v) \le \mu\} \gg_n \mu^j (M_1 \cdot M_2 \cdots M_j)^{-1}. \tag{5.10}$$

For the upper bound (see e.g. Davenport [Dav58]). For convenience we include the short argument here. Define positive integers $\nu_1, \dots, \nu_j$ such that

$$2^{\nu_i - 1} \leq \frac{2\mu}{M_i} < 2^{\nu_i}. \tag{5.11}$$

Hence, $\nu_1 \geq \nu_2 \dots \geq \nu_1$. Consider another element $v' \in \Lambda$ with $F(v') \leq \mu$ and $v' = m_1' a_1 + \dots m_j' a_j$ with some $m_j' \in \mathbb{Z}$. Assuming for the moment that $m_l \equiv m_l' \mod 2^{nu_l}$, $l = 1, \dots, j$, and let i denote the largest index i such that $m_i \neq m_i'$. Then $x \stackrel{\text{def}}{=} 2^{-\nu_i}(v - v')$ is an element of Λ and linearly independent of $a_1, \dots, a_{i-1}$. Thus we conclude $F(x) \geq M_i$. But

$$F(x) = 2^{-\nu_i} F(v - v') \leq 2^{-\nu_i}(F(v) + F(v')) \leq 2^{-\nu_i} 2\mu < M_i$$

by (5.11). The contradiction shows that there is at most one lattice point in Λ such the coordinates $m_1, \dots, m_j$ lie in the same residue classes to the moduli $2^{\nu_1}, 2^{\nu_2}, \dots, 2^{\nu_j}$ respectively. Hence the number of lattice points $N(\mu)$ in (5.10) is bounded from above by the number of all residue classes, i.e. by $2^{\nu_1} 2^{\nu_2} \dots 2^{\nu_j} \leq (4\mu)^j (M_1 \dots M_j)^{-1}$. This shows the upper bound in (5.7). $\square$

Lemma 5.5. *(Davenport* [Dav58]*, Minkowski) Let $\Lambda = A\mathbb{Z}^n$ and $\Lambda' = A'\mathbb{Z}^n$ denote dual lattices of rank n, i.e.* $\det(\Lambda)\det(\Lambda') = 1$ *and $\langle Au, A'v \rangle = \langle u, v \rangle$ for any vectors $u, v \in \mathbb{Z}^n$ or $AA'^T = I_n$. Let $M_j, j = 1, \dots d$, and $N_j, j = 1, \dots n$, denote the successive minima of Λ resp. of Λ' with respect to Euclidean norm $\|\cdot\|$. Let $c_-(n) \stackrel{\text{def}}{=} (n!)^{-1} 2^n / \omega_n$ and $c_+ \stackrel{\text{def}}{=} 2^n/\omega_n$ where ω_n denotes the volume of the unit n-ball. Then we have*

$$\begin{aligned} c_-(n)\det(\Lambda) &\leq M_1 \dots M_n \leq c_+(n)\det(\Lambda), && (5.12)\\ c_-(n)\det(\Lambda') &\leq N_1 \dots N_n \leq c_+(n)\det(\Lambda') && (5.13)\\ 1 &\leq M_j N_{n+1-j} \leq c_+(n), \quad j = 1, \dots, n && (5.14) \end{aligned}$$

Proof. The first two inequalities are special cases of Minkowski's inequality for the successive minima of arbitrary norms F on Λ

$$\det(\Lambda)\frac{2^n}{n!V} \leq M_1 \dots M_n \leq \frac{2^n}{V}\det(\Lambda), \tag{5.15}$$

where V denotes the volume of the convex body $F < 1$. By the unimodularity and the choice of F we have $V = \omega_n$. The left hand side of (5.14) follows by the definition of the successive minima M_j and N_{n+1-j} and the existence of u resp. v in sublattices of rank j resp. $n + 1 - j$ of Λ resp. Λ' such that $\langle u, v \rangle \neq 0$ that is $\|u\|\|v\| \geq |\langle u, v \rangle| \geq 1$. The right hand side follows by Minkowski's inequalities for both successive minima. For details, see (Davenport (1958),[Dav58], Lemma 2) $\square$

5.1. **Conjugate lattices.** Now we shall apply the previous results to the special norms $H_t(m,\bar m)^{1/2}$ in the theta series (4.40). Here and below writing (a,b), for $a\in\mathbb{R}^d$, $b\in\mathbb{R}^d$, means that $(a,b)\in\mathbb{R}^{2d}$ and the coordinates of (a,b) are the coordinates of the vectors a and b in the corresponding order, that is, $(a,b)=(a_1,a_2,\dots,a_d,b_1,b_2,\dots,b_d)$. In the following we shall use methods from the Geometry of Numbers for lattices in $\mathbb{R}^n$ with $n=2d$. Recall that H_t is a positive quadratic form on $\mathbb{R}^d\times\mathbb{R}^d$ given by

$$H_t[m,\bar m]=r^2Q_+^{-1}[m-t'Q\bar m]+r^2Q_+[\bar m],\quad t'\stackrel{\text{def}}{=}\frac{2}{\pi}t,\quad t\in\mathbb{R},\quad r\geq 1. \tag{5.16}$$

Let $\langle\cdot,\cdot\rangle_t$ denote the associated scalarproduct $\langle l,l\rangle_t\stackrel{\text{def}}{=}H_t(l), l\in\mathbb{R}^d\times R^d$. We shall rewrite $H_t[m,\bar m]$ by means of elements of $SL(2d,\mathbb{R})$ and lattices in $\mathbb{R}^{2d}$ as follows. Introduce for $(m,\bar m)\in\mathbb{Z}^{2d}$, $u\stackrel{\text{def}}{=}Q_+^{-1/2}m\in\mathbb{R}^d$ and $v\stackrel{\text{def}}{=}Q_+^{-1/2}Q\bar m=SQ_+^{1/2}\bar m\in\mathbb{R}^d$, where $S=QQ_+^{-1}$ is a reflection. Note that S, Q and Q_+ commute. Let $T\in SL(2d,\mathbb{Z})$ be the permutation matrix which reorders the $2d$ coordinates $(u,\,v)=(u_1,\dots,u_d,\,v_1,\dots,v_d)\in\mathbb{R}^{2\,d}$ into d pairs given by

$$T(u,v)\stackrel{\text{def}}{=}(\eta_1,\dots,\eta_d),\quad\text{where}\quad \eta_j\stackrel{\text{def}}{=}(u_j,v_j),\quad j=1,\dots,d. \tag{5.17}$$

Let Λ_Q denote the lattice of rank $2d$ of all vectors $T(u,\,v)$, $\quad(m,\bar m)\in\mathbb{Z}^{2d}$, i.e.

$$\Lambda_Q\stackrel{\text{def}}{=}T\,A_Q\mathbb{Z}^{2\,d},\quad\text{where}\quad A_Q\stackrel{\text{def}}{=}\begin{pmatrix} Q_+^{-1/2} & O_d\\ O_d & SQ_+^{1/2}\end{pmatrix}. \tag{5.18}$$

Note that $\det(\Lambda_Q)=1$. Thus we may write $H_t[m,\bar m]=r^2\|u-t'v\|^2+r^{-2}\|v\|^2=\sum_{j=1}^d\|d_r\,u_t\,\eta_j\|^2$, where

$$d_r\stackrel{\text{def}}{=}\begin{pmatrix} r & 0\\ 0 & r^{-1}\end{pmatrix},\qquad u_t\stackrel{\text{def}}{=}\begin{pmatrix} 1 & -t'\\ 0 & 1\end{pmatrix}. \tag{5.19}$$

Here $D\stackrel{\text{def}}{=}\{d_r, r>0\}$, and $U\stackrel{\text{def}}{=}\{u_t, t\in\mathbb{R}\}$, are quasi-geodesic and unipotent one-parameter subgroups of $SL(2,\mathbb{R})$. We shall consider the diagonal representation of $g\in SL(2,\mathbb{R})$ on $(\mathbb{R}^2)^d=\mathbb{R}^{2d}$ defined by

$$g(\eta_1,\dots,\eta_d)\stackrel{\text{def}}{=}(g\,\eta_1,\dots,g\,\eta_d),\quad g\in SL(2,\mathbb{R}),. \tag{5.20}$$

Denote in particular the representation of $g=d_r$ and $g=u_t$ on $(\mathbb{R}^2)^d$ by the same symbols g, d_r and u_t. Thus we may rewrite (5.16) as follows

$$\begin{aligned} H_t(m,\bar m) &= r^2\|Q_+^{-1/2}m-t'SQ_+^{1/2}\bar m\|^2+r^{-2}\|Q_+^{1/2}\bar m\|^2=\|d_r\,u_t\,T\,A_Q(m,\bar m)^T\|^2\\ &= \|d_ru_t\eta\|^2,\qquad\text{where}\quad \eta\stackrel{\text{def}}{=}(\eta_1,\dots,\eta_d)\in(\mathbb{R}^2)^d. \end{aligned} \tag{5.21}$$

Note that in view of (5.5) the α_l-characteristic of the $2d$-dimensional lattice $\Lambda_t\stackrel{\text{def}}{=}d_ru_t\Lambda_Q$ may be expressed by the reciprocal of the minimal volume of a l-dimensional

sublattice, generated by elements, say $a_j = d_r u_t T A_Q l_j$, where l_j is a basis of an l-dimensional sublattice of $\mathbb{Z}^{2d}$. Thus

$$\alpha_l(\Lambda_t) = \det\Big(\langle a_j, a_k\rangle, j,k = 1,\ldots,d\Big)^{-1/2}. \tag{5.22}$$

By construction $\langle a_j, a_k\rangle = \langle l_j, l_k\rangle_t$.
Based on the special symplectic structure of our $2d$-dimensional lattices we may sharpen Lemma 5.3 and Lemma 5.5 as follows.

Lemma 5.6. *Let $M_1,\ldots,M_{2d}$ the successive minima of $\Lambda_t = d_r u_t \Lambda_Q$ with respect to the norm $\|\cdot\|$. Then we have for any t and $\mu \ge 1$*

$$M_j M_{2d+1-j} \asymp_d 1, \quad j = 1,\ldots,d, \tag{5.23}$$
$$r^{-1} q_0^{1/2} \le M_1 \le \ldots \le M_d \ll_d 1 \le M_{j+d}, \quad j = 1,\ldots d, \tag{5.24}$$
$$\#\{v \in \Lambda_t : \quad \|v\| < \mu\} \ll_d \mu^{2d}(M_1 \cdot M_2 \cdots M_d)^{-1} \asymp_d \mu^{2d}\alpha_d(\Lambda_t). \tag{5.25}$$

Corollary 5.7. *Notice that*

$$\alpha(\Lambda_t) = \max\{\alpha_j(\Lambda_t) \,:\, j = 1,\ldots,2d\} \asymp_d \alpha_d(\Lambda_t). \tag{5.26}$$

Proof. First we prove (5.23). Let $J \overset{\text{def}}{=} k_{\pi/2}$ denote the imaginary unit in $SL(2,\mathbb{Z})$. Then we have the following obvious relations

$$\begin{aligned} J k_\theta &= k_\theta J, \quad d_r J = J d_{r^{-1}}, \quad u_t^T J = J u_{-t}, \quad \text{and} \\ d_r u_t &= u_{t r^2} d_r. \end{aligned} \tag{5.27}$$

For $B \in GL(d,\mathbb{R})$ define $J_B \in SL(2d,\mathbb{R})$ as:

$$J_B \overset{\text{def}}{=} \begin{pmatrix} O_d & -B \\ B & O_d \end{pmatrix}. \tag{5.28}$$

Using the reflection $S \in Gl(d,\mathbb{R})$ defined above define

$$\Lambda_t = d_r u_t \Lambda_Q = d_r u_t T A_Q \mathbb{Z}^{2d} \quad \text{and} \quad \Lambda_t' \overset{\text{def}}{=} d_r u_t T A_Q J_S \mathbb{Z}^{2d}. \tag{5.29}$$

Note that these lattices are adjoint with repect to the diagonal action of J on $(\mathbb{R}^2)^d$. To see this let for any $N, N' \in \mathbb{Z}^{2d}$

$$w(N) \overset{\text{def}}{=} d_r u_t T A_Q N \in \Lambda_t, \quad w_J(N') \overset{\text{def}}{=} J d_r u_t T A_Q(-J_S) N' \in \Lambda_t'.$$

Using (5.27) together with $JT = T J_I$, $T \in O(2d)$, $A_Q J_I A_Q = J_S$ and $J_S^2 = -Id$ we obtain

$$\langle w(N), w_J(N')\rangle = \langle N, N'\rangle, \tag{5.30}$$

for any $N, N' \in \mathbb{Z}^{2d}$. Hence Λ_t and Λ_t' are dual in the sense of Lemma 5.5. We claim that they have identical successive minima. To this end note that with

$N = (m, \bar m)^T \in \mathbb{Z}^{2d}$,

$$\begin{aligned}\|w(N)\| &= \|(Q_+^{-1/2} m - t' Q_+^{-1/2} Q \bar m\,,\, Q_+^{1/2} \bar m)\| \\ &= \|(Q_+^{-1/2} S m - t' Q_+^{-1/2} Q S \bar m, Q_+^{1/2} S \bar m)\| = \|w((-J_S) J_I N)\| \\ &= \|w'(J_I N)\|, \end{aligned} \tag{5.31}$$

where we have used $(-J_S)J_I(m,\bar m)^T = (Sm, S\bar m)^T$ and in the last equality $\|Jw\| = \|w\|$. Since $J_I\mathbb{Z}^{2d} = \mathbb{Z}^{2d}$, the equation (5.31) implies that the successive minima of Λ_t and Λ_t' are identical and by Lemma 5.5 we conclude $M_j M_{2d+1-j} \asymp_d 1$ for $j = 1, \dots, d$.

Proof of (5.24). Since $M_d \le M_{d+1}$ and $1 \ll_d M_d M_{d+1} \ll_d 1$ we obtain $M_j \le M_d \ll_d 1 \ll_d M_{d+1} \le M_{j+d}$ for all $j = 1, \dots, d$. The inequality $M_1 \ge q_0^{1/2} r^{-1}$ immediately follows by definition of the first successive minimum and $r^{-1}\|Q_+^{1/2} n\| \ge q_0^{1/2} r^{-1}$.

Proof of (5.25). Let $2d \ge j \ge 1$ denote the maximal integer such that $M_j \le b$. Then Lemma 5.3 yields with $\mu > 1$
$\#\{v \in \Lambda_t : \quad \|v\| < \mu\} \le (4\mu)^j (M_1 \dots M_j)^{-1} \ll_d \mu^{2d} (M_1 \dots M_d)^{-1}$,
since in the case $j > d$ we have $M_j \ge \dots \ge M_d \gg_d 1$ and in case $j < d$ we obtain $\mu < M_{j+1} \le \dots \le M_d \ll_d 1$. The last inequality in (5.25) follows by Lemma 5.3 and the corollary is immediate by the last arguments (with $\mu = 1$). $\square$

For arbitrary t and for small t the following bounds which are independent of the Diophantine properties of Q hold.

Lemma 5.8. *Let $q_0^{1/2} \le N_{Q,1} \le N_{Q,2} \le \dots N_{Q,d} \le q^{1/2}$ denote the successive minima of the lattice $\Lambda \overset{\text{def}}{=} Q_+^{1/2}\mathbb{Z}^d$. We have*

$$\sup_t \alpha_d(d_s u_t \Lambda_Q) \ll_d \varphi_Q(s), \quad \textit{where} \tag{5.32}$$

$$\varphi_Q(s) \overset{\text{def}}{=} s^d |\det Q|^{-1/2} \prod_{j\,:\,N_{Q,j} > s} \Big(\frac{N_{Q,j}}{s}\Big)^2, \ s > 0. \quad \textit{Note that}$$

$$\varphi_Q(s) \ll_d s^{-d} |\det Q|^{1/2}, \quad \textit{if} \quad |s| \le q_0^{1/2}, \quad \textit{and} \tag{5.33}$$

$$\ll_d s^d |\det Q|^{-1/2}, \quad \textit{if} \quad |s| \ge q^{1/2} \tag{5.34}$$

For small t we get

$$\alpha_d(d_r u_t \Lambda_Q) \ll_d |\det Q|^{1/2} (r^{-1} + |t\,r|)^d, \quad \textit{if} \quad q_0^{1/2} |tr| \ge 1, \tag{5.35}$$

$$\alpha_d(d_r u_t \Lambda_Q) \ll_d |\det Q|^{-1/2} |t\,r|^{-d}, \qquad \textit{if} \quad q^{1/2}|tr| \le 1. \tag{5.36}$$

Proof. By Lemma 5.5, relations (5.7) and (5.25) with $\mu = 1/2$ we conclude that

$$\alpha_d(\Lambda_t) \asymp_d (M_1 \dots M_d)^{-1} \asymp_d \#\{v \in \Lambda_t \,:\, \|v\|^2 < 1/4\}. \tag{5.37}$$

Recall that $\|v\|^2 < 1/4$ with $v \in \Lambda_t$ (see (5.16)) may be written as

$$H_t[m, \bar m] = r^2 Q_+^{-1}[m - t' Q \bar m] + r^{-2} Q_+[\bar m] < 1/4. \tag{5.38}$$

Proof of (5.32). Assume that $s \in (q_0^{1/2}, q^{1/2}]$. Then (5.38) implies $\|Q_+^{1/2}\bar m\| < s/2$ which by Lemma 5.3, inequality (5.7), has at most $c_d \prod_{j\,:\,N_{Q,j}\le s}\left(\frac{s}{N_{Q,j}}\right)$ solutions. Similarly (5.38) implies by triangle inequality that for fixed $\bar m$ (in (5.38)) the number of pairs $(m, \bar m)$ for which (5.38) holds is bounded by the number of elements v in the dual lattice $\Lambda' = Q^{-1/2}\mathbb{Z}^d$ to Λ such that $s\|v\| < 1$. Since the successive minima for this dual lattice are determined by Lemma 5.5, we may use Lemma 5.3, (inequality (5.7)), again to determine the upper bound $c_d' \prod_{j\,:\,N_{Q,j}\ge s}\left(\frac{N_{Q,j}}{s^{-1}}\right)$ for this number as well. The product of both numbers yields the desired bound for the number of solutions of (5.38). Finally, using (5.37) and Minkowskis Theorem 5.5, (inequality (5.13) for non unimodular lattices), i.e. $\prod_j N_{Q,j} \asymp_d |\det Q|^{-1/2}$, shows the desired bound $\varphi_Q(s)$ in (5.32) of Lemma 5.8. The inequalities (5.33), (5.34) for $s \ge q^{1/2}$ and $s \le q_0^{1/2}$ are immediate by these arguments.

Proof of (5.35). Assume $q_0^{1/2}|t'\,r| \ge 1$ and $q_0 \ge 1$. If $m = 0$ we conclude that $q_0\|\bar m\|^2 \le |t'\,r|^2\|Q_+^{1/2}\bar m\|^2 < 1/4$. Hence $\bar m = 0$. For any fixed $m \ne 0$ and $r^2\|Q_+^{-1/2}m - t'Q_+^{+1/2}\,S\,\bar m\|^2 < 1/4$ the triangle inequality implies that there is at most one element $\bar m \in \mathbb{Z}^d$ with $r\|Q_+^{-1/2}m - t'Q_+^{+1/2}\,S\,\bar m\| < 1/2$. Furthermore, we get $|\|Q_+^{-1/2}m\| - 1/(2\,r)| \le \|t'Q_+^{+1/2}\,\bar m\|$ for that pair $(m, \bar m)$. This implies

$$1/4 > H_t(m, \bar m) \ge r^{-2}\|Q_+^{+1/2}\,\bar m\|^2 \ge \Big|\|Q_+^{-1/2}m\| - 1/(2\,r)\Big|^2/|r\,t'|^2$$

and hence $\Big|\|Q_+^{-1/2}m\| - 1/(2\,r)\Big| \le |r\,t'|/2$. Thus

$$\#\{v \in \Lambda_t\,:\,\|v\|^2 < 1/4\} \ll_d (r^{-1} + |r\,t'|)^d\,|\det Q|^{1/2}.$$

Proof of (5.36). Similarly, the equation (5.38) implies by triangle inequality that

$$|\|Q_+^{-1/2}m\| - \|t'Q_+^{1/2}\,S\,\bar m\|| \le (2r)^{-1} < (2q^{1/2})^{-1}. \tag{5.39}$$

Again (5.38) implies $|t'r|\,r^{-1}\|Q_+^{1/2}\bar m\| \le |t'r|1/2 \le (4q^{1/2})^{-1}$ using $2q^{1/2}|t'\,r| \le 1$. This leads to a contradiction unless $m = 0$. Hence, the possible solutions for $\bar m$ in (5.39) satisfy $\|Q_+^{1/2}\bar m\| \le |2t'r|^{-1}$ which as in the proof of (5.32) has at most $c_d|\det Q|^{-1/2}|2t'r|^{-d}$ solutions which completes the proof of (5.36) in view of (5.37).

□

5.2. **Approximation by compact subgroups.** Later on we need to average over powers of the α_d-characteristic of $2d$-dimensional lattices Λ_t in Section 6. In order to use harmonic analysis tools we shall approximate the subgroup U of $SL(2,\mathbb{R})$ in (5.20) locally by a compact subgroup $K = SO(2) = \{k\theta\,:\,\theta \in [0, 2\pi]\}$ with elements

$$k_\theta \stackrel{\text{def}}{=} \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}. \tag{5.40}$$

Lemma 5.9. *We have for $t \in [-2, 2]$, $r > 1$ and any $2d$-dimensional lattice Λ in $(\mathbb{R}^2)^d$ with diagonal action of d_r, u_s and k_θ*

$$\begin{aligned} \alpha_j(d_r u_t \Lambda) \;&\ll_d\; \alpha_j(d_r k_\theta \Lambda), \quad j = 1, \ldots, 2d, \quad \textit{where} \\ \theta \;&=\; \arcsin\big(t\,(1+t^2)^{-1/2}\big), \quad \textit{or, equivalently,} \quad t = \tan\theta. \end{aligned} \tag{5.41}$$

Proof. Let $m, \overline{m} \in \mathbb{R}$. Note that $\overline{m} + tm = (1+t^2)\,\overline{m} + t\,(m - t\overline{m})$, which implies that

$$\begin{aligned} (\overline{m} + tm)^2 \;&\le\; 2(1+t^2)^2\,(\overline{m})^2 + 2\,t^2\,(m - t\overline{m})^2, \;\text{ and} \\ r^2\,(m - t\overline{m})^2 + r^{-2}\,(\overline{m} + tm)^2 \;&\le\; 2\,(1+t^2)^2\,\big(r^2\,(m - t\overline{m})^2 + r^{-2}\overline{m}^2\big), \end{aligned} \tag{5.42}$$

for $r \ge 1$. By definition of θ in (5.42) we have

$$|\theta| \le c^* \overset{\text{def}}{=} \arcsin(2/\sqrt{5}), \quad \cos\theta = (1+t^2)^{-1/2}, \quad \sin\theta = t\,(1+t^2)^{-1/2}. \tag{5.43}$$

Dividing (5.42) by $2\,(1+t^2)^2$, using (5.40) and (5.42), we get with $\eta \overset{\text{def}}{=} (\overline{m}, m)$

$$\|d_r\, u_t\, \eta\|^2 \;\ge\; \|d_r\, k_\theta\, \eta\|^2\,(1+t^2)^{-1}/2 \;\ge\; \|d_r\, k_\theta\, \eta\|^2/10,$$

for any $\eta \in \Lambda$. Applying Lemma 5.2 to the norms $F_1(\eta) = \|d_r\, u_t\, \eta\|$ and $F_2(\eta) = \|d_r\, k_\theta\, \eta\|$ and using Lemma 5.3 we conclude the proof of Lemma 5.2. $\square$

5.3. **Irrational and Diophantine lattices.** We consider Diophantine approximations of $A \overset{\text{def}}{=} t\,Q$ by integral matrices using, see (2.12), with approximation error

$$\delta_{A;s} \overset{\text{def}}{=} \min\Big\{\|M - m\,A\| \,:\, m \in \mathbb{Z}, 0 < |m| \le R,\, M \in \mathrm{M(d,\mathbb{Z})}\Big\}, \; \mathrm{R} \ge 1. \tag{5.44}$$

Note that by Lemma 5.8 the quantity $\beta_{t;r} \overset{\text{def}}{=} \alpha_d(\Lambda_t) r^{-d} |\mathrm{det} Q|^{1/2}$ is uniformly bounded (in t), that is $\beta_{t;r} \ll_d 1$ for $r > q^{1/2}$. Larger values of $\beta_{t;r}$ enforce smaller values of the rational approximation error $\delta_{tQ;R}$.

Lemma 5.10. *Assume that $q_0 \ge 1$. Then we have for all t and $r \ge 1$, writing $R_t \overset{\text{def}}{=} \beta_{t;r}^{-1}$*

$$\delta_{tQ;R_t} < R_t r^{-2} d^{1/2} \tag{5.45}$$

Note that this bound is nontrivial for $\beta_{t;r} \gg_d r^{-2}$ only.

Before proving (5.47), we shall state some important consequences.

Corollary 5.11. *Consider any interval $[T_-, T]$ with $T_- \in (0,1)$ and $T > 2$.*
i) If Q is irrational then

$$\lim_{r\to\infty} \sup_{T_- \le t \le T} \alpha_d(\Lambda_t)\, r^{-d} = 0. \tag{5.46}$$

ii) If Q is Diophantine of type (κ, A), where $1 > \kappa > 0$, $A > 0$, that is,

$$\inf_{t\in[1,2]} \delta_{tQ,R} > A\,R^{-\kappa}, \quad \textit{for all} \quad R \ge 1, \tag{5.47}$$

then

$$\sup_{T_- \le t \le T} \alpha_d(\Lambda_t) r^{-d} |\mathrm{det} Q|^{-1/2} \le A^{-1} \max\left\{T_-^{\kappa+1}, T^{\kappa}\right\} r^{-2+2\kappa}. \tag{5.48}$$

Proof. i) Assume that there is an $\varepsilon > 0$ and sequence r_j, t_j such that $\lim_j r_j = \infty$ and $\beta_{t_j;r_j} > \varepsilon$. Passing to a subsequence we may assume that $\lim_j t_j = t$, $t \in I \stackrel{\text{def}}{=} [T_-, T]$. Thus (5.47) yields $\lim_j r_j^2 \beta_{t_j;R_j^*} = 0$, $R_j^* \stackrel{\text{def}}{=} \beta_{t_j;r_j}^{-1} < \varepsilon^{-1}$, or $\lim_j \|M_j - t_j m_j Q\| = 0$ for some $M_j \in M(d, \mathbb{Z})$ and $m_j \in Z$ with $|m_j| \le \varepsilon^{-1}$. Obviously both, $\|M_j\|$ and $|m_j|$ are bounded. Hence there exist integral elements M, m and an infinite subsequence j' of j with $M_{j'} = M$, $m_{j'} = m$ and by construction $\lim_{j'} t_{j'} = t$. These limit values satisfy $\|M - mtQ\| = 0$, that is Q is rational. The last arguments are similar to those in [Göt04], p. 224, Lemma 3.11, which yields an alternative proof of (5.46) when combined with (5.6).

ii) Write $\beta_{I;r} \stackrel{\text{def}}{=} \sup_{t \in I} \beta_{t;r}$. Since $R \to (\delta_{tQ;R})^{-1}$ is nondecreasing, we obtain for every $t \in I = [T_-, T]$ with $\beta_{t;r} > r^{-2}$ using (5.47) that

$$\beta_{t;r} \le r^{-2} (\delta_{tQ,R_t})^{-1} \le r^{-2} (\delta_{tQ,r^2})^{-1} \le A^{-1} r^{-2+2\kappa} \tag{5.49}$$

In case that $\beta_{t;r} \le r^{-2}$ this bound obviously hold. Hence we conclude (5.48) as claimed. □

Proof. of Lemma 5.10. For fixed t we rewrite the α_d-characteristic using the quadratic form H_t as in (5.22) as follows. Let $D_{rQ} \stackrel{\text{def}}{=} \begin{pmatrix} rQ_+^{-1/2} & 0 \\ 0 & r^{-1} Q_+^{1/2} \end{pmatrix}$ and $U_{tQ} \stackrel{\text{def}}{=} \begin{pmatrix} I_d & -t' Q \\ 0 & I_d \end{pmatrix}$ denote elements of $\mathrm{SL}(2d, \mathbb{R})$, where $t' \stackrel{\text{def}}{=} \frac{2}{\pi} t$ Write D_r for D_{rQ} with $Q = I_d$. Then $H_t(l) = \|D_Q D_r U_{tQ} l\|^2 = \|d_r u_t T A_Q l\|^2$ using the notations of (5.17) – (5.21). Introduce the $2d$-dimensional lattice $\Omega_t \stackrel{\text{def}}{=} D_r U_{tQ} \mathbb{Z}^{2d}$. The α-characteristic of the lattices $\Lambda_t = d_r u_t \Lambda_Q$, where $\Lambda_Q = T A_Q \mathbb{Z}^{2d}$, may be rewritten in view of(5.22) as $\alpha_d(\Lambda_t) = \alpha_d(D_Q \Omega_t) = \|\bar{w}_1 \wedge \ldots \wedge \bar{w}_d\|$, by means of d-vectors $\bar{w}_j \in D_Q \Omega_t$. Here we used the standard Euclidean norm on the exterior product $E_d \stackrel{\text{def}}{=} \wedge^d \mathbb{R}^{2d}$. If $q_0 \ge 1$ the minimal eigenvalue of the dth exterior power $\wedge^d D_Q$ of D_Q on E_d is given by $|\mathrm{det} Q|^{-1/2}$. Thus for any $w \in E_d$ we have $\|\wedge^d D_Q w\| \ge |\mathrm{det} Q|^{-1/2} \|w\|$ and we obtain

$$\alpha_d(\Lambda_t) = \alpha_d(D_Q \Omega_t) \le |\mathrm{det} Q|^{1/2} \alpha_d(\Omega_t). \tag{5.50}$$

Write $\alpha_d(\Omega_t) = \|D_r U_{tQ} l_1 \wedge \ldots \wedge D_r U_{tQ} l_d\|^{-1}$, where $l_j = (m_j, n_j)^T \in \mathbb{Z}^{2d}$ is a basis of a d-dimensional sublattice of $\mathbb{Z}^{2d}$. Let L denote the d-dimensional subspace of $\mathbb{R}^{2d}$ generated by $D_r U_{tQ} l_j, j = 1, \ldots, d$. Introduce the $d \times d$ integer matrices N_t and M_t with columns $n_1, \ldots, n_d$ resp. $m_1, \ldots, m_d$. We claim that

$$\alpha_d(\Omega_t) > r^{-(d-2)} \quad \text{implies} \quad r^2 > |\det(N_t)| > 0. \tag{5.51}$$

Assuming this fact for the moment, we may parametrize the subspace L as follows. Let W denote the linear map $v \to (M N^{-1} v, v)^T$ from $\mathbb{R}^d$ to $\mathbb{R}^{2d}$. Then $L = D_r U_{tQ} W R^d$.

The form $\langle v, v\rangle_t \stackrel{\text{def}}{=} \|D_r U_{tQ} W v\|^2$ is positive definite and defines a scalar product on $\mathbb{R}^d$, (similar to H_t in (5.16)). Choosing an orthonormal basis $v_1, \dots, v_d$ of $\mathbb{R}^d$ with respect to $\langle\cdot,\cdot\rangle$ which diagonalizes $\langle v, v\rangle_t$ we may write $n_j = Av_j$, $j = 1, \dots, d$, for some element $A \in \mathrm{GL}(d, \mathbb{R})$ such that $\det A = \det N_t$. By definition we have

$$\begin{aligned}
\alpha_d(\Omega_t)^{-1} &= \|D_r U_{tQ} W n_1 \wedge \ldots \wedge D_r U_{tQ} W n_d\| = \det\Big(\langle n_j, n_k\rangle_t,\ j, k = 1, \dots, d\Big)^{1/2} \\
&= |\det A| \det\Big(\langle v_j, v_k\rangle_t,\ j, k = 1, \dots, d\Big)^{1/2} \\
&= |\det N_t| \prod_{j=1}^{d} \|D_r U_{tQ} W v_j\|. \qquad (5.52)
\end{aligned}$$

By assumption $R_t = \beta_{t;r}^{-1} < r^2$. Since $\alpha_d(\Lambda_t)^{-1} = |\det Q|^{-1/2} r^{-d} \beta_{t;r}^{-1}$, this means that by (5.50) that $\alpha_d(\Omega_t)^{-1} \le r^{-d} R_t$. Furthermore, by (5.52)

$$\alpha_d(\Omega_t)^{-1} \ge |\det N_t| \max_j \|D_r U_{tQ} W v_j\| r^{-(d-1)}, \qquad (5.53)$$

since $\|D_r U_{tQ} W v_j\| \ge r^{-1}$ for all j. Using $\|D_r U_{tQ} W v_j\| \ge r\,\|(M_t N_t^{-1} - tQ)v_j\|$ for all j, we conclude that

$$r^{-d} R_t \ge \alpha_d(\Omega_t)^{-1} \ge |\det N_t| r^{-(d-2)} d^{-1/2} \|(M_t N_t^{-1} - tQ)\|. \qquad (5.54)$$

Since $(\det N_t)\, N_t^{-1}$ is an integral matrix, (5.53) and (5.54) together imply

$$\min\{\|\bar{M} - mtQ\| \,:\, 0 < |m| \le R_t,\ m, \bar{M} \text{ integral}\} < d^{1/2} R_t r^{-2},$$

which proves (5.45).

It remains to show (5.51). Assume that $\operatorname{rank}(N_t) = d-k$. Let accordingly $n_1, \dots, n_k$, say, denote k vectors which are linearly dependent of the independent vectors $n_{k+1}, \dots, n_d$. Let $w_j \stackrel{\text{def}}{=} D_r U_{tQ} l_j$. We may rewrite $\beta \stackrel{\text{def}}{=} \|w_1 \wedge \ldots \wedge w_d\|$ replacing $w_1, \dots, w_k$ by linear combinations with $w_{k+1}, \dots, w_d$ such that the new first k vectors w_j are of the form $w_j = (r\bar{m}_j, 0)^T$, $j = 1, \dots, k$, where $\bar{m}_j \in \mathbb{Z}^d$. Let $\bar{l}_j$ denote the corresponding vectors in $\mathbb{Z}^{2d}$ such that $w_j = D_r U_{tQ} \bar{l}_j$. Combine the column vectors $\bar{l}_j, j = 1, \dots d$ into $d \times d$ matrices $\bar{M}$ and $\bar{N}$ (for the first and second coordinates) as above. Express β^2 as a sum of squares of the $\binom{2d}{d}$ coordinates of $w_1 \wedge \ldots \wedge w_d$, say $\beta_{J,K}$, indexed by pairs of subsets $J \subset \{1, \dots, d\}$ and $K \subset \{d+1, \dots, 2d\}$ with $|J| + |K| = d$. Then

$$|\beta_{J,K}| = \left|\det\begin{pmatrix} r(\bar{M} - tQ\bar{N})_J \\ r^{-1}\bar{N}_K \end{pmatrix}\right| = \left|\det\begin{pmatrix} r\bar{M}_{J,k} & r\,(\bar{M} - tQ\tilde{N})_{J,d-k} \\ 0 & r^{-1}\,\bar{N}_{K,d-k} \end{pmatrix}\right|, \qquad (5.55)$$

where $\bar{M}_J$, resp. $\bar{M}_K$ denotes the submatrix of $\bar{M}$ with row numbers J resp. K. Similarly, let $\bar{M}_{J,k}$ resp. $\bar{M}_{J,d-k}$ denote the submatrix of $\bar{M}_J$ with column numbers $1, \dots, k$, respectively $k+1, \dots, d$. Similar notations are used for the other occurring block matrices. Note that by the rank assumption $\beta_{J,K} = 0$ unless $|K| \le d-k$. Thus we may choose a subset K with $|K| = d - k$ such that $\det\bar{N}_{K,d-k} \ne 0$. Similarly

we may choose a subset J as well with $|J| = k$ and $\det \bar{M}_{J,k} \neq 0$. Since the $k \times k$-matrix $\bar{M}_{J,k}$ thus obtained is invertible we may evaluate the determinant $|\beta_{J,K}|$ via $|\beta_{J,K}| = |r^k (\det \bar{M}_{J,k}) r^{-(d-k)} (\det \bar{N}_{K,d-k})|$. Hence by (5.55)

$$\beta \geq |\beta_{J,K}| \geq r^{-(d-2k)} (\det \bar{N}_{K,d-k}) (\det \bar{M}_{J,k}) \geq r^{-(d-2k)}, \tag{5.56}$$

which yields a contradiction to $\alpha_d(\Omega_t)^{-1} = \beta < r^{d-2}$, unless $k = 0$, thus proving the claim in (5.51).
Thus we may assume $k = 0$ or $\det N_t \neq 0$ and here (5.56) implies $\beta \geq r^{-d} \det N_t$. Furthermore, by definition $\beta < r^{-d} R_t$, which finally yields $|\det N_t| < R_t$, thus proving the remaining upper bound for $|\det N_t|$ in (5.51). □

6. Averages Along Along Translates of Orbits of K

6.1. **Application of Geometry of Numbers.** In order to estimate the error terms $I_{1,\pm}$ in (4.4) we need in view of Lemma 4.1 and Lemma 4.4 and (4.39) an estimate of

$$\int_{|t|>1/r} \psi(r,t)^{1/2} g_w(t) dt,$$

where g_w denotes the smoothed indicator function of $[a, b]$, see (3.18), which is bounded from above by $\exp\{-c_1|wt|^\kappa\}, |t| \geq 1$ with $0 < \kappa < 1$ and $0 < w < 1$, see (4.3), and ψ denotes the theta-series of (4.40). We shall start with bounds of this integral over an interval $I \stackrel{\text{def}}{=} [t_0 - 2, t_0 + 2], t_0 \in \mathbb{R}$ of fixed length.

The results obtained by geometry of numbers in Lemma 4.5 and (5.25) in Lemma 5.6 will be used to bound this integral. Introduce the maximum value over I of the α_d-characteristic for the lattices $\Lambda_t = d_r u_t \Lambda_Q$, (using the notations in (5.18) and (5.19)), as well as for the lattices $\Lambda_{t,Q} \stackrel{\text{def}}{=} d_{r_Q} u_t \Lambda_Q$, where $r_Q \stackrel{\text{def}}{=} q^{1/2}$, via

$$\gamma_{I,\beta}(r) \stackrel{\text{def}}{=} \sup\Big\{ \Big(r^{-d} \alpha_d(d_r u_t \Lambda_Q)\Big)^{\frac{1}{2}-\beta} \;:\; t \in I\Big\}. \tag{6.1}$$

Here $\gamma_{I,\beta}(r)$ depends on the Diophantine properties of an irrational Q and tends to zero for growing r by Lemma 5.11. Using the notations of section 5.2 the following Lemma holds.

Lemma 6.1. *Let $I = [t_0 - 2, t_0 + 2]$, $r > r_Q$ and choose $0 < \beta < 1/2$ such that $\beta d > 2$, $\beta < 1/2$ and let $d \geq 5$. Then we have with $\widehat{g}_I \stackrel{\text{def}}{=} \max\{|\widehat{g}_w(t)| \,:\, t \in I\}$*

$$\begin{aligned} &\sup_{v \in \mathbb{R}^d} \int_I \Big| \theta_v(t) \widehat{g}_w(t) \Big| dt \\ &\ll \; \widehat{g}_I |\det Q|^{-1/4} r^{d-\beta d} \gamma_{I,\beta}(r) \max_{j \in J} \int_{-\pi}^{\pi} \alpha_d(d_{r_0} k_\theta \Lambda_{Q,s_j})^\beta \frac{d\theta}{2\pi}, \end{aligned} \tag{6.2}$$

where $r_0 \stackrel{\text{def}}{=} \frac{r}{r_Q}$ and J denotes set of points $s_j \stackrel{\text{def}}{=} t_0 - 2 + j\,\frac{2}{r_Q} \in I$, $j = 1, 2, \ldots$.

Proof. Estimate $|\widehat{g}_w|$ by its maximum $\widehat{g}_I$ on I and use Lemma 4.4 to bound $\theta_v(t)$ by $\psi(r,t)^{1/2}$. In view of Lemma 5.5 with $\varepsilon = 1$ together with (5.25) for $\mu = d^{1/2}$, we have

$$\begin{aligned} \psi(r,t) &\leq \#\{w \in \Lambda_t \,:\, \|w\|_\infty < 1\} \leq \#\{w \in \Lambda_t \,:\, \|w\| < d^{1/2}\} \\ &\ll_d d^{2d} \alpha_d(\Lambda_t) \ll_d r^{d/2-\beta\, d} \gamma_{I,\beta}(r)\, \alpha_d(\Lambda_t)^\beta. \end{aligned} \tag{6.3}$$

Similarly, estimating $|\widehat{g}_w|$ by its maximum $\widehat{g}_I$ on I and using Lemma 4.4 to bound $\theta_v(t)$ by $|\det Q|^{-1/4} r^{d/2} \psi(r,t)^{1/2}$, we get

$$R \overset{\text{def}}{=} \sup_{v \in \mathbb{R}^d} \int_I \Big|\theta_v(t)\, \widehat{g}_w(t)\Big| dt \ll_d \widehat{g}_I\, |\det Q|^{-1/4} r^{d-d\beta} \gamma_{I,\beta}(r) \int_I \alpha_d(\Lambda_t)^\beta\, dt. \tag{6.4}$$

Using (5.27), we have $d_r\, u_t = d_r\, u_{t-s_j}\, u_{s_j} = d_{r_0}\, d_s\, d_{r_Q}\, u_{s_j}$, where $s \overset{\text{def}}{=} (t-s_j)\, r_Q^2$. Changing variables from t to s we obtain in terms of the lattices $\Lambda_{Q,s}$

$$\begin{aligned} \int_I \alpha_d(\Lambda_t)^\beta\, dt &\ll_d \sum_{j \in J} \int_{[t_{j-1},t_j]} \alpha_d(d_{r_0}\, u_s\, d_{r_Q}\, u_{s_j}\, \Lambda_Q)^\beta\, dt \\ &\ll_d \max_{s_j \in J} \int_{-2}^{2} \alpha_d(d_{r_0}\, u_s\, \Lambda_{Q,s_j})^\beta\, ds. \end{aligned} \tag{6.5}$$

Finally, we estimate the last average, using Lemma 5.9, by the average over the group K. Change variables $\theta(s) = \arcsin\big(s/(1+s^2)^{1/2}\big)$, $s \in [-2,2]$. Note that $|\theta| \leq c^* = \arcsin(2/\sqrt{5}) < 1.2$. With $ds = \sqrt{1+s^2}^{\,-1} d\theta$ we get (see (5.42)) by completing the integration over θ from the subinterval $[-c^*, c^*]$ to the arc $[-\pi, \pi]$

$$\int_{-2}^{2} \alpha_d(d_{r_0}\, u_s\, \Lambda_{Q,s_j})^\beta\, ds \ll \int_{-2}^{2} \alpha_d(d_r\, k_{\theta(s)}\, \Lambda_{Q,s_j})^\beta ds \ll \int_{-\pi}^{\pi} \alpha_d(d_r\, k_\theta\, \Lambda_{Q,s_j})^\beta\, \frac{d\theta}{2\pi}.$$

This together with inequalities (6.4) and (6.5) completes the proof. $\square$

In the following paragraphs we shall develop explicit bounds for averages over the group K of type $\int_K \alpha_d(d_r\, k\Lambda)^\beta\, dk$.

6.2. **Operators A_g and functions τ_λ on $SL(2,\mathbf{R})$.** Let $G = SL(2,\mathbb{R})$. We consider two subgroups of G:

$$K = SO(2) = \left\{ k(\theta) = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \,:\, 0 \leq \theta < 2\pi \right\}$$

and

$$T = \left\{ \begin{pmatrix} a & b \\ 0 & a^{-1} \end{pmatrix} : a > 0, b \in \mathbb{R} \right\}.$$

According to the Iwasawa decomposition, any $g \in G$ can be uniquely represented as a product of elements from K and T, that is

$$g = k(g)t(g), \quad k(g) \in K, t(g) \in T.$$

Now let

$$d_a \stackrel{\text{def}}{=} \left\{ \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix} : a > 0 \right\} \text{ and } D^+ = \{d_a : a \geq 1\}.$$

According to the Cartan decomposition,

$$G = KD^+K, \ g = k_1(g)d(g)k_2(g), \ g \in G, k_1(g), k_2(g) \in K, \ d(g) \in D^+.$$

In this decomposition $d(g)$ is determined by g, and if $g \notin K$ then $k_1(g)$ and $k_2(g)$ are also determined by g. It is clear that $\|g\| = \|d(g)\|$. Since d_a is the conjugate of $d_{a^{-1}}$ by $k(\pi/2)$, we have that $g^{-1} \in KgK$ or equivalently, $d(g) = d(g^{-1})$ for any $g \in G$. Therefore, $\|g\| = \|g^{-1}\|, g \in G$.

We say that a function f on G is *left K-invariant* (resp. *right K-invariant*, resp. *bi-K-invariant* if $f(Kg) = f(g)$ (resp. $f(gK) = f(g)$, resp. $f(KgK) = f(g)$). Any bi-K-invariant function on G is completely determined by its restriction to D^+. Hence for any bi-K-invariant function f on G, one can find a function $\hat{f}$ on $[1, \infty)$ such that $f(g) = \hat{f}(\|g\|), g \in G$.

For any $\lambda \in \mathbb{R}$ we define a character x_λ of T by

$$\chi_\lambda \begin{pmatrix} a & b \\ 0 & a^{-1} \end{pmatrix} = a^{-\lambda},$$

and the function $\varphi_\lambda : G \to \mathbb{R}^+$ by

$$\varphi_\lambda(g) = \chi_\lambda(t(g)), \quad g \in G.$$

The function φ_λ has the property

$$\varphi_\lambda(kgt) = \chi_\lambda(t)\varphi_\lambda(t), \quad g \in G, k \in K, t \in T, \tag{6.6}$$

and it is completely determined by this property and by the condition $\varphi_\lambda(1) = 1$.

For $g \in G$ and a continuous action of G on a topological space X, we define the operator A_g on the space of continuous functions on X by

$$(A_g f)(x) = \int_K f(gkx) d\sigma(k), \quad x \in X, \tag{6.7}$$

where σ denotes the normalized Haar measure on K or, using the parametrization of K, by

$$(A_g f)(x) = \frac{1}{2\pi} \int_0^{2\pi} f(gk(\theta)x) d\theta, \quad x \in X.$$

The operator A_g is a linear map into the space of K-invariant function on X. If $X = G$ and G acts on itself by left translations, then A_g commutes with right translations.

From these two remarks, or using a direct computation, we get that $A_g\varphi_\lambda$ has the property (6.6). Hence φ_λ is an eigenfunction for A_g with the eigenvalue

$$\tau_\lambda(g) \stackrel{\text{def}}{=} (A_g\varphi_\lambda)(1) = \int_K \varphi_\lambda(gk)d\sigma(k) = \int_K \chi_\lambda(t(g(k))d\sigma(k). \tag{6.8}$$

We see from (6.8) that τ_λ is obtained from φ_λ by averaging over right translations by elements of K. But φ_λ is left K-invariant and A_g commutes with right translations. Hence this function τ_λ is bi-K-invariant and it is an eigenfunction for A_g with the eigenvalue $\tau_\lambda(g)$, that is

$$A_g\tau_\lambda = \tau_\lambda(g)\tau_\lambda \quad \text{or} \quad (A_g\tau_\lambda)(h) = \tau_\lambda(g)\tau_\lambda(h), h \in G. \tag{6.9}$$

We have that

$$\varphi_\lambda(g) = \|ge_1\|^{-\lambda}, \quad g \in G, e_1 = (1,0), \tag{6.10}$$

where $\|\cdot\|$ denotes the usual Euclidean norm on $\mathbb{R}^2$. Indeed

$$\varphi_\lambda(g) = \chi_\lambda(t(g)) = \|t(g)e_1\|^{-\lambda} = \|k(g)t(g)e_1\|^{-\lambda} = \|ge_1\|^{-\lambda}.$$

From (6.8) and (6.10) we get

$$\begin{aligned} \tau_\lambda(g) &= \int_K \|gke_1\|^{-\lambda}d\sigma(k) = \frac{1}{2\pi}\int_0^{2\pi} \|gk(\theta)e_1\|^{-\lambda}d\theta \qquad (6.11)\\ &= \frac{1}{2\pi}\int_0^{2\pi} \|g(\cos\theta,\sin\theta)\|^{-\lambda}d\theta = \int_{S_1} \|gu\|^{-\lambda}d\ell(u), \end{aligned}$$

where S^1 is the unit circle in $\mathbb{R}^2$ and ℓ denotes the normalized rotation invariant measure on S^1. One can easily see that $\|gu\|^{-2}, g \in G, u \in S^1$, is equal to the Jacobian at u of the diffeomorphism $v \mapsto gv/\|v\|$ of S^1 onto S^1. On the other hand, it follows from the change of variables formula that

$$\int_M J_f^\lambda = \int_M J_{f^{-1}}^{1-\lambda}, \quad \lambda \in \mathbb{R},$$

where $f : M \to M$ is a diffeomorphism of a compact differentiable manifold M and J_f (resp. $J_{f^{-1}}$) denotes the Jacobian of f (resp. f^{-1}). Now using (6.11) we get

$$\tau_\lambda(g) = \tau_{2-\lambda}(g^{-1}) = \tau_{2-\lambda}(g), g \in G, \lambda \in \mathbb{R}. \tag{6.12}$$

The second equality in (6.12) is true because τ_λ is bi-K-invariant and $g^{-1} \in KgK$. Since, obviously, $\tau_0(g) = 1$, it follows that

$$\tau_2(g) = \tau_0(g) = 1. \tag{6.13}$$

Since $t^{-\lambda}$ is a strictly convex function of λ for any $t > 0, t \neq 1$, it follows from (6.11) that $\tau_\lambda(g)$ is a strictly convex function of λ for any $g \in K$. From this, (6.12) and (6.13) we deduce that

$$\tau_\eta(g) < \tau_\lambda(g) \quad \text{for any} \quad g \notin K \text{ and } 1 \leq \eta < \lambda,$$

$$\tau_\eta(g) < 1 \text{ and } \tau_\lambda(g) > 1 \quad \text{for any } g \notin K,\ 0 < \eta < 2, \lambda > 2, \tag{6.14}$$

and

$$\tau_\eta(g) < \tau_\lambda(g) \quad \text{for any } g \notin K, \lambda \geq 2,\ 0 < \eta < \lambda. \tag{6.15}$$

Since the function $\tau_\lambda(g)$ is bi-K-invariant, it depends only on the norm $\|g\|$ of g. We can write

$$\tau_\lambda(g) = \hat{\tau}_\lambda(\|g\|), \quad g \in G, \tag{6.16}$$

where

$$\begin{aligned} \hat{\tau}_\lambda(a) \quad &\stackrel{\text{def}}{=} \tau_\lambda(d_a) = \int_K \|d_a k e_1\|^{-\lambda} d\sigma(k) \qquad (6.17) \\ = \quad &\frac{1}{2\pi} \int_0^{2\pi} \frac{d\theta}{(a^2 \cos^2\theta + a^{-2}\sin^2\theta)^{\lambda/2}}, \quad a \geq 1. \end{aligned}$$

It follows from (6.9) and the definition of A_g that

$$\int_K \hat{\tau}_\lambda(\|gkd_a\|)\, d\sigma(k) = \tau_\lambda(g)\hat{\tau}_\lambda(a), \quad g \in G, a \geq 1. \tag{6.18}$$

Since $\|g\| = \|g^{-1}\|$ for all $g \in G$,

$$\frac{a}{\|g\|} \leq \|gkd_a\| \leq a\|g\|$$

for all $k \in K$ and $g \in G$. From this, (6.14) and (6.18) we deduce that, for any $\lambda > 2$, the continuous function $\hat{\tau}_\lambda(a), a \geq 1$, does not have a local maximum. Hence $\hat{\tau}_\lambda$ is strictly increasing for all $\lambda > 2$ or, equivalently,

$$\tau_\lambda(g) < \tau_\lambda(h) \quad \text{if } \|g\| < \|h\|,\ g, h \in G,\ \lambda > 2. \tag{6.19}$$

It follows from (6.12) and (6.17) that

$$\hat{\tau}_\lambda(a) = \hat{\tau}_{2-\lambda}(a) = \frac{1}{2\pi} \int_0^{2\pi} (a^2\cos^2\theta + a^{-2}\sin^2\theta)^{\frac{\lambda}{2}-1} d\theta. \tag{6.20}$$

Since $a^2\cos^2\theta \leq a^2\cos^2\theta + a^{-2}\sin^2\theta \leq a^2$, we deduce from (6.20) the following estimates

$$c(\lambda)a^{\lambda-2} \leq \hat{\tau}_\lambda(a) \leq a^{\lambda-2},\ a \geq 1,\ \lambda \geq 2, \tag{6.21}$$

where

$$c(\lambda) = \frac{1}{2\pi} \int_0^{2\pi} (\cos\theta)^{\lambda-2} = \frac{(2\lambda-3)}{2^{\lambda-2}\Gamma(\lambda-1)^2}.$$

It also follows from (6.20) that, for any $\lambda > 2$, the ratio $\frac{\hat{\tau}_\lambda(a)}{a^{\lambda-2}}$ is a strictly decreasing function of $a \geq 1$ and

$$\lim_{a\to\infty} \frac{\hat{\tau}_\lambda(a)}{a^{\lambda-2}} = c(\lambda).$$

Lemma 6.2. *Let $g \in G, g \notin K, \lambda > 2, 0 < \eta < \lambda, b \geq 0, Q > 1$, and let f be a left K-invariant positive continuous function on G. Assume that*

$$A_g f \leq \tau_\lambda(g) f + b\tau_\eta \tag{6.22}$$

and that

$$f(yh) \leq Qf(h) \quad \textit{if} \quad h \in G, y \in G \textit{ and } \|y\| \leq \|g\|. \tag{6.23}$$

Then for all $h \in G$,

$$(A_h f)(1) = \int_K f(hk)\, d\sigma(k) \leq s\tau_\lambda(h),$$

where

$$s = Q\left[f(1) + \frac{b}{\tau_\lambda(g) - \tau_\eta(g)}\right]. \tag{6.24}$$

Proof. Let

$$f_K(h) \overset{\text{def}}{=} \int_K f(hk)\, d\sigma(k), \quad h \in G.$$

Since A_g commutes with right translations, and τ_η is right K-invariant, it follows from (6.22) that $A_g f_K \leq \tau_\lambda(g) f_K + b\tau_\eta$. If h and y are as in (6.23), then $f(yhk) \leq Qf(hk)$ for every $k \in K$ and therefore $f_K(yh) \leq Qf_K(h)$. On the other hand, it is clear that

$$f_K(h) = (A_h f_K)(1) = (A_h f)(1).$$

Thus we can replace f by f_K and assume that f is bi-K-invariant. Then we have to prove that $f \leq s\tau_\lambda$. Assume the contrary. Then $f(h) > s'\tau_\lambda(h)$ for some $h \in G$ and $s' > s$. In view of (6.15) and (6.24), $s' > s \geq Qf(1)$. From this, (6.19) and (6.23) we get that $\|h\| > \|g\|$ and

$$f(yh) > \frac{s'}{Q}\, \tau_\lambda(yh) \quad \text{if } \|y\| \leq \|g\| \text{ and } \|yh\| \leq \|h\|. \tag{6.25}$$

□

Using the Cartan decomposition, we see that any $x \in G$ with $\frac{\|h\|}{\|g\|} \leq \|x\| \leq \|h\|$ can be written as $x = k_1 y h k_2$ where $k_1, k_2 \in K, \|y\| \leq \|g\|$ and $\|yh\| \leq \|h\|$. But the functions f and τ_λ are bi-K-invariant. Therefore we can get from (6.25) that

$$f(x) > \frac{s'}{Q}\, \tau_\lambda(x) \text{ if } \frac{\|h\|}{\|g\|} \leq \|x\| \leq \|h\|. \tag{6.26}$$

Let

$$\begin{aligned} a_\eta &\stackrel{\text{def}}{=} \frac{b}{\tau_\lambda(g) - \tau_\eta(g)},\ a_\lambda \stackrel{\text{def}}{=} \frac{s'}{Q} > f(1) + \frac{b}{\tau_\lambda(g) - \tau_\eta(g)}, \text{ and} \\ \omega &= f - a_\lambda \tau_\lambda + a_\eta \tau_\eta. \end{aligned}$$

Then, in view of (6.9) and (6.22),

$$\begin{aligned} &A_g\omega - \tau_\lambda(g)\omega = A_g(f - a_\lambda\tau_\lambda + a_\eta\tau_\eta) - \tau_\lambda(g)(f - a_\lambda\tau_\lambda + a_\eta\tau_\eta) \\ &= [A(g)f - \tau_\lambda(g)f] - a_\lambda[A_g\tau_\lambda - \tau_\lambda(g)\tau_\lambda] + a_\eta\,[A_g\tau_\eta - \tau_\lambda(g)\tau_\eta] \\ &\leq b\tau_\eta + a_\eta\,[\tau_\eta(g)\tau_\eta - \tau_\lambda(g)\tau_\eta] = 0. \end{aligned} \tag{6.27}$$

Since $\tau_\lambda(1) = \tau_\eta(1) = 1$, we have

$$\omega(1) = f(1) - a_\lambda + a_\eta < 0. \tag{6.28}$$

It follows from (6.15) that $a_\eta \geq 0$. From this, (6.24) and (6.26) we get that

$$\begin{aligned} \omega(x) &= f(x) - a_\lambda\tau_\lambda(x) + a_\eta\tau_\eta(x) \geq f(x) - a_\lambda\tau_\lambda(x) \\ &> \left(\frac{s'}{Q} - a_\lambda\right)\tau_\lambda(x) = 0 \text{ if } \frac{\|h\|}{\|g\|} \leq \|x\| \leq \|h\|. \end{aligned} \tag{6.29}$$

Let $v \in G, \|v\| \leq \|h\|$, be a point where the continuous function ω attains its minimum on the set $\{x \in G : \|x\| \leq \|h\|\}$. It follows from (6.28) and (6.29) that

$$\omega(v) < 0 \text{ and } \|v\| \leq \frac{\|h\|}{\|g\|}.$$

Then, since $\tau_\lambda(g) > 1$ and $\|gkv\| \leq \|g\|\|v\|$ for all $k \in K$,

$$(A_g\omega)(v) = \int_K \omega(gkv)d\sigma(k) \geq \omega(v) > \tau_\lambda(g)\omega(v).$$

We get a contradiction with (6.27).

As a special case ($\eta = 2$ and $b = 0$) of Lemma 6.2, we have the following

Lemma 6.3. *Let $g \in G, g \notin K, \lambda > 2, Q > 1$, and let f be a left K-invariant positive continuous function on G satisfying the inequality (6.23). Assume that*

$$A_g f \leq \tau_\lambda(g) f.$$

Then for all $h \in G$,

$$(A_h f)(1) = \int_K f(hk)d\sigma(k) \le Qf(1)\tau_\lambda(h).$$

Lemma 6.4. *Let $g \in G, g \notin K, 2 < \lambda < \mu, Q > 1, M > 1, n \in \mathbb{N}^+$, and let $f_i, 0 \le i \le n$, be left K-invariant positive continuous functions on G. We denote $\min\{i, n-i\}$ by $\bar{i}$ and $\sum_{0 \le i \le n} f_i$ by f. Assume that*

$$f_i(yh) \le Qf_i(h) \quad \text{if } 0 \le i \le n, h \in G, y \in G \text{ and } \|y\| \le \|g\|,$$

$$A_g f_i \le \tau_\lambda(g) f_i + M \max_{0 < j \le \bar{i}} \sqrt{f_{i-j} f_{i+j}}, \quad 0 \le i \le n, \tag{6.30}$$

in particular,

$$A_g f_0 \le \tau_\lambda(g) f_0 \text{ and } A_g f_n \le \tau_\lambda(g) f_n.$$

Then there is a constant $B = B(g, \lambda, \mu, Q, M, n)$ such that for all $h \in G$,

$$(A_h f)(1) = \int_K f(hk)d\sigma(k) \le Bf(1)\tau_\mu(h). \tag{6.31}$$

Proof. For any $0 < \varepsilon \le 1$ and $0 \le i \le n$ we define

$$f_{i,\varepsilon} = \varepsilon^{q(i)} f_i \quad \text{where } q(i) \overset{\text{def}}{=} i(n-i).$$

It follows from (6.30) that for all $i, 0 \le i \le n$,

$$\begin{aligned} &A_g f_{i,\varepsilon} = \varepsilon^{q(i)} A_g f_i \le \varepsilon^{q(i)} \tau_\lambda(g) f_i \\ &+\varepsilon^{q(i)} M \max_{0 < j \le \bar{i}} \sqrt{\varepsilon^{-q(i-j)} f_{i-j,\varepsilon} \varepsilon^{-q(i+j)} f_{i+j,\varepsilon}} \\ &= \tau_\lambda(g) f_{i,\varepsilon} + M \max_{0 < j \le \bar{i}} \sqrt{\varepsilon^{q(i) - \frac{1}{2}[q(i-j)+q(i+j)]} f_{i-j,\varepsilon} f_{i+j,\varepsilon}}. \end{aligned}$$

Direct computation shows that

$$q(i) - \frac{1}{2}[q(i-j) + q(i+j)] = j^2.$$

Hence for all $i, 0 \le i \le n$,

$$A_g f_{i,\varepsilon} \le \tau_\lambda(g) f_{i,\varepsilon} + \varepsilon M \max_{0 < j \le \bar{i}} \sqrt{f_{i-j,\varepsilon} f_{i+j,\varepsilon}}. \tag{6.32}$$

Let $f_\varepsilon \overset{\text{def}}{=} \sum_{0 \le i \le n} f_{i,\varepsilon}$. Summing (6.32) over all $i, 0 \le i \le n$, and using the inequalities $f_\varepsilon > \sqrt{f_{i-j,\varepsilon}, f_{i+j,\varepsilon}}$, $1 \le i \le n-1, 0 < j \le \bar{i}$, we get

$$\begin{aligned} A_g f_\varepsilon = \sum_{0 \le i \le n} A_g f_{i,\varepsilon} &\le \tau_\lambda(g) f_\varepsilon + \varepsilon M(n-1) f_\varepsilon \\ &= [\tau_\lambda(g) + \varepsilon M(n-1)] f_\varepsilon. \end{aligned} \tag{6.33}$$

Let

$$\varepsilon_0 = \min\left\{1, \frac{\tau_\mu(g) - \tau_\lambda(g)}{M(n-1}\right\}.$$

Then it follows from (6.33) that

$$A_g f_{\varepsilon_0} \le \tau_\mu(g) f_{\varepsilon_0}.$$

Now we apply Lemma 6.3 to f_{ε_0} and get that for all $h \in G$,

$$\begin{aligned}(A_h f)(1) &< \varepsilon_0^{-n^2}(A_h f_{\varepsilon_0})(1) \le \varepsilon_0^{-n^2} f_{\varepsilon_0}(1)\tau_\mu(h) \\ &\le \varepsilon_0^{-n^2} Q f(1)\tau_\mu(h).\end{aligned}$$

Hence (6.31) is true with $B = \varepsilon_0^{-n^2} Q$. $\square$

Proposition 6.5. *Let $g \in G, g \notin K, d \in \mathbb{N}^+, Q > 1, M > 1$. For every $0 \le i \le 2d$, let $\lambda_i \ge 2$ and let f_i be a left K-invariant positive continuous function on G. We denote $\min\{i, 2d-i\}$ by $\bar{i}$ and $\sum_{0 \le i \le 2d} f_i$ by f. Assume that*

$$\lambda_d > \lambda_i \quad \text{for any } i \ne d.$$

$$f_i(yh) \le Q f_i(h) \quad \text{if } 0 \le i \le 2d, h \in G, y \in G \text{ and } \|y\| \le \|g\|, \tag{6.34}$$

$$A_g f_i \le \tau_{\lambda_i}(g) f_i + M \max_{0 < j \le \bar{i}} \sqrt{f_{i-j} f_{i+j}}, \quad 0 \le i \le 2d, \tag{6.35}$$

in particular,

$$A_g f_0 \le \tau_{\lambda_0}(g) f_0 \quad \text{and} \quad A_g f_{2d} \le \tau_{\lambda_{2d}}(g) f_{2d}.$$

Then, using the notation $\ll$ (which until the end of the proof of this proposition means that the left hand side is bounded from above by the right hand side multiplied by a constant which depends on $g, \lambda_0, \dots, \lambda_{2d}, Q$ and M, and does not depend on $f_0, \dots, f_{2d}$), we have that

(a) For all $h \in G$ and $0 \le i \le 2d, i \ne d$,

$$(A_h f_i)(1) = \int f_i(hk) d\sigma(k) \ll f(1)\tau_\eta(h),$$

where

$$\eta = \lambda_d - 3^{-(d+1)}(\lambda_d - \eta') < \lambda_d \quad \text{and} \quad \eta' = \max\{\lambda_i : 0 \le i \le 2d, i \ne d\}. \tag{6.36}$$

(b) For all $h \in G$,

$$(A_h f_d)(1) = \int_K f_d(hk) d\sigma(k) \ll f(1)\tau_{\lambda_d}(h).$$

(c) For all $h \in G$,

$$(A_h f)(1) = \int_K f(hk) d\sigma(k) \ll f(1)\|h\|^{\lambda_d - 2}.$$

Proof. (a) Let

$$f_{i,K}(h) \stackrel{\text{def}}{=} \int\limits_K f_i(hk)d\sigma(k), h \in G.$$

It follows from the Cauchy–Schwartz inequality that

$$\begin{aligned}
&\int \sqrt{f_{i-j}(hk)f_{i+j}(hk)}\ d\sigma(k)\\
&\leq \sqrt{\int f_{i-j}(hk)\sigma(k)}\sqrt{\int f_{i+j}(hk)d\sigma(k)}\\
&= \sqrt{f_{i-j,K}(h)f_{i+j,K}(h)}.
\end{aligned}$$

Hence

$$\begin{aligned}
&\int \max_{0<j\leq\bar{i}} \sqrt{f_{i-j}(hk)f_{i+j}(hk)}\ d\sigma(k)\\
&\leq \sum_{0<j\leq\bar{i}} \int \sqrt{f_{i-j}(hk)f_{i+j}(hk)}\ d\sigma(k)\\
&\leq \sum_{0<j\leq\bar{i}} \sqrt{f_{i-j,K}(h)f_{i+j,K}(h)} \leq d \max_{0<j\leq\bar{i}} \sqrt{f_{i-j,K}(h)f_{i+j,K}(h)}.
\end{aligned}$$

On the other hand,

$$(A_g f_{i,K})(h) = \int\limits_K (A_g f_i)(hk)d\sigma(k)$$

and, according to (6.35),

$$(A_g f_i)(hk) \leq \tau_{\lambda_i}(g)f_i(hk) + M \max_{0<j\leq\bar{i}} \sqrt{f_{i-j}(hk)f_{i+j}(hk)}.$$

Therefore

$$A_g f_{i,K} \leq \tau_{\lambda_i}(g)f_{i,K} + dM \max_{0<j\leq\bar{i}} \sqrt{f_{i-j}f_{i+j}}.$$

But

$$f_{i,K}(h) = (A_h f_{i,K})(1) = (A_h f_i)(1)$$

and, as easily follows from (6.34),

$$f_{i,K}(yh) \leq Q f_{i,K}(yh)$$

if $h \in G, y \in G$ and $\|y\| \leq \|g\|$. Thus, replacing f_i by $f_{i,K}$ and M by dM, we can assume that the functions f_i are bi-K-invariant. Then we have to prove that

$$f_1 \ll f(1)\tau_\eta \quad \text{for all } 0 \leq i \leq 2d, i \neq d. \tag{6.37}$$

□

Let, as in (6.36), $\eta' = \max\{\lambda_i : 0 \le i \le 2d, i \ne d\}$. We define $\mu_i, 0 \le i \le 2d$, by

$$\mu_d = \lambda_d + 3^{-(d+1)}(\lambda_d - \eta') \quad \text{and} \tag{6.38}$$

$$\mu_i = \mu_d - 3^{-\bar{i}}(\lambda_d - \eta'), 0 \le i \le 2d,\ i \ne d. \tag{6.39}$$

Since, in view of (6.15), $\tau_{\lambda_i}(g) \le \tau_{\mu_d}(g)$, it follows from (6.15) and Lemma 6.4 that

$$f_i \ll f(1)\tau_{\mu_d}, \quad 0 \le i \le 2d. \tag{6.40}$$

One can easily check that $\eta > \mu_i > \lambda_i \ge 2$ and therefore $\tau_\eta \ge \tau_{\mu_i}$ for all $0 \le i \le 2d, i \ne d$. Thus, to prove (6.37), it is enough to show that

$$f_i \ll f(1)\tau_{\mu_i} \quad \text{for all } 0 \le i \le 2d, i \ne d. \tag{6.41}$$

We will prove (6.41) for $i \le d-1$ using induction in i; the proof in the case $i \ge d+1$ is similar. In view of (6.15), $\tau_{\mu_0}(g) > \tau_{\lambda_0}(g)$. Then for $i = 0$ it is enough to use Lemma 6.3. Let $1 \le m \le d-1$ and assume that (6.41) is proved for all $i < m$ Then, in view of (6.40), for $0 < j \le m$,

$$\begin{aligned} \sqrt{f_{m-j}f_{m+j}} &\ll f(1)\sqrt{\tau_{\mu_{m-j}}\tau_{\mu_d}} \\ &\le f(1)\sqrt{\tau_{\mu_{m-1}}\tau_{\mu_d}} \ll f(1)\tau_{(\mu_{m-1}+\mu_d)/2}. \end{aligned} \tag{6.42}$$

(The second inequality in (6.42) follows from (6.15) and (6.39), and the third one follows from (6.16) and (6.21).)

Combining (6.35) and (6.37) we get

$$A_g f_m \le \tau_{\lambda_m}(g) f_m + C f(1)\tau_{(\mu_{m-1}+\mu_d)/2},$$

where

$$C \ll 1.$$

On the other hand, as follows from (6.38) and (6.39), $\lambda_m < \mu_m$ and $(\mu_{m-1}+\mu_d)/2 < \mu_m$. Now, to prove that $f_m \ll f(1)\tau_{\mu_m}$, it remains to apply Lemma 6.2 (and use again ((6.15)).

(b) As in the proof of (a), we can assume that the functions f_i are bi-K-invariant. Then we get from (6.35) and (6.37) that

$$A_g f_d \le \tau_{\lambda_d} f_d + D f(1)\tau_\eta,$$

where $D \ll 1$. But $\eta < \lambda_d$. Therefore, Lemma 6.2 implies that $f_d \ll f(1)\tau_{\lambda_d}$ which proves (b).

(c) Follows from (a), (b), (6.15), (6.16) and (6.21).

6.3. **Quasinorms and representations of $SL(2,\mathbb{R})$.** We say that a continuous function $v \mapsto |v|$ on a real topological vector space V is a *quasinorm* if it has the properties:

(i) $|v| \geq 0$ and $|v| = 0$ if and only if $v = 0$;
(ii) $|\lambda v| = |\lambda||v|$ for all $\lambda \in \mathbb{R}$ and $v \in V$.

If V is finite dimensional, then any two quasinorms on V are equivalent in the sense that their ratio lies between two positive constants.

Lemma 6.6. *Let ρ be a (continuous) representation of $G = SL(2,\mathbb{R})$ in a real topological vector space V, let $|\cdot|$ be a K-invariant quasinorm on V, and let $v \in V, v \neq 0$, be an eigenvector for T corresponding to the character $\chi_{-r}, r \in \mathbb{R}$, that is*

$$\rho \begin{pmatrix} a & b \\ 0 & a^{-1} \end{pmatrix} v = a^r v.$$

Then for any $g \in G$ and $\beta \in \mathbb{R}$,

$$|\rho(g)v|^{-\beta} = \varphi_{\beta r}(g)|v|^{-\beta} \tag{6.43}$$

and

$$\int\limits_K \frac{d\sigma(k)}{|\rho(gk)v|^{\beta}} = \tau_{\beta r}(g)/|v|^{\beta} \tag{6.44}$$

Proof. Using the K-invariance of $|\cdot|$ we get that

$$\begin{aligned} |\rho(g)v|^{-\beta} &= |\rho(k(g))\rho(t(g))v|^{-\beta} \\ &= |\rho(t(g))v|^{-\beta} \\ &= |\chi_{-r}(t(g))v|^{-\beta} = \chi_{\beta r}(t(g))|v|^{-\beta} = \varphi_{\beta r}(g)|v|^{-\beta}. \end{aligned}$$

The equality (6.44) follows from (6.43) and from the definition of $\tau_{\beta r}(g)$. □

Let $\|z\|$ denote the norm of $z \in \mathbb{C}^2$ corresponding to the standard Hermitian inner product on $\mathbb{C}^2$, that is

$$\|z\|^2 = \|x\|^2 + \|y\|^2 \quad \text{where } z = x + iy, x, y \in \mathbb{R}^2.$$

Lemma 6.7. *For any $z \in \mathbb{C}^2, z \neq 0, g \in G$ and $\beta > 0$,*

$$F(z) = F_{g,\beta}(z) \stackrel{\text{def}}{=} \|z\|^{\beta} \int\limits_K \frac{d\sigma(k)}{\|gkz\|^{\beta}} \leq \tau_{\beta}(g).$$

Proof. Since the measure σ on K is translation invariant,

$$F(kz) = F(z) \text{ for any } k \in K. \tag{6.45}$$

Also, for all $\lambda \in \mathbb{C}$, $\lambda \neq 0$, and $z \in \mathbb{C}^2, z \neq 0$,

$$F(\lambda z) = F(z) \tag{6.46}$$

because $\|\lambda v\| = |\lambda|\|v\|, v \in \mathbb{C}^2$, and because $G = SL(2,\mathbb{R})$ acts linearly on $\mathbb{C}^2$. Any nonzero vector $x \in \mathbb{R}^2$ can be represented as $x = \lambda k e_1, \lambda \in \mathbb{R}, k \in K, e_1 = (1,0)$. Then, using (6.11) from section 6.2, we get from (6.45) and (6.46) that

$$F(x) = F(e_1) = \tau_\beta(g) \text{ for all } x \in \mathbb{R}^2, x \neq 0. \tag{6.47}$$

Let now $z = x + iy, x, y \in \mathbb{R}^2, z \neq 0$. We write $e^{i\theta}z = x_\theta + iy_\theta, x_\theta, y_\theta \in \mathbb{R}^2$. Then $\frac{\|x_\theta\|}{\|y_\theta\|}$ is a continuous function of θ with values in $\mathbb{R} \cup \{\infty\}$. But $e^{i\pi/2}z = iz = -y + ix$ and therefore $\frac{\|x_{\pi/2}\|}{\|y_{\pi/2}\|} = \left(\frac{\|x_0\|}{\|y_0\|}\right)^{-1}$. Hence there exists θ such that $\|x_\theta\| = \|y_\theta\|$. Replacing then z by $e^{i\theta}z$ and using (6.46) we can assume that $\|x_\theta\| = \|y_\theta\|$. Now using the convexity of the function $t \to t^{-\beta/2}$, $t > 0$, and the equality (6.47) we get that

$$\begin{aligned}
&\int_K \frac{d\sigma(k)}{\|gkz\|^\beta} = \int_K \frac{d\sigma(k)}{(\|gkx\|^2 + \|gky\|^2)^{\beta/2}} \\
&\leq 2^{-\beta/2} \cdot \frac{1}{2}\left[\int_K \frac{d\sigma(k)}{\|gkz\|^\beta} + \int_K \frac{d\sigma(k)}{\|gky\|^\beta}\right] \\
&= 2^{-\beta/2} \cdot \frac{1}{2}\left[\frac{\tau_\beta(g)}{\|x\|_\beta} + \frac{\tau_\beta(g)}{\|y\|^\beta}\right] = 2^{-\beta/2}\tau_\beta(g)\frac{1}{\|x\|^\beta} \\
&= 2^{-\beta/2}\tau_\beta(g) \cdot \frac{1}{\|z\|^\beta \cdot 2^{-\beta/2}} = \frac{\tau_\beta(g)}{\|z\|^\beta}.
\end{aligned}$$

Clearly (6.48) implies (6.43). □

Let $\mathcal{P}_n$ denote the $(m+1)$-dimensional space of real homogeneous polynomials in two variables of degree m, and let ψ_m denote the regular representation of $G = SL(2,\mathbb{R})$ in $\mathcal{P}_m : (\psi_m(g)P)(x) = P(g^{-1}x), g \in G, x \in \mathbb{R}^2, P \in \mathcal{P}_m$. It is well known that, the representation ρ_m is irreducible for any m and that any irreducible finite dimensional representation of G is isomorphic to ψ_m for some m. Recall that all finite dimensional representations ρ of G are fully reducible, i.e. ρ can be decomposed into the direct sum of irreducible representations. We define

$$I(\rho) = \{m \in \mathbb{N}^+ : \psi_m \text{ is isomorphic to a subrepresentation of } \rho\}.$$

Proposition 6.8. *Let ρ be a representation of $G = SL(2,\mathbb{R})$ in a finite dimensional space W. Then there exists a K-invariant quasinorm $|\cdot| = |\cdot|_\rho$ on W such that for any $w \in W, w \neq 0$, $g \in G$ and $\beta > 0$,*

$$\int_K \frac{d\sigma(k)}{|\rho(gk)w|^\beta} \leq \max_{m \in I(\rho)}\{\tau_{\beta m}(g)\}\frac{1}{|w|^\beta}.$$

Proof. Let $W = \bigoplus_{i=1}^{n} W_i$ be the decomposition of W into the direct sum of $\rho(G)$ - irreducible subspaces, and let $\pi_i : W \to W_i$ denote the natural projection. Suppose that we constructed for each i a K-invariant quasinorm $|\cdot|_i = |\cdot|_{\rho_i}$ on W_i such that for any $w \in W_i, w \neq 0, g \in G$, and $\beta > 0$,

$$\int_K \frac{d\sigma(k)}{|\rho_i(gk)w|^\beta} \leq \tau_{\beta m(i)}(g)\ \frac{1}{|w|_1^\beta}, \tag{6.48}$$

where ρ_i denotes the restriction of ρ to W_i and $m(i) \in I(\rho)$ is defined by the condition that $\psi_{m(i)}$ is isomorphic to ρ_i. Then we define $|w| = |w|_\rho$ by

$$|w| = \max_{1 \leq i \leq n} |\pi_i(w)|_i,\ w \in W. \tag{6.49}$$

Clearly $|\cdot|_\rho$ is a K-invariant quasinorm. Let us fix now $w \in W, w \neq 0$, and choose i such that $|w| = |\pi_i(w)|_i$. Then

$$\begin{aligned}
\int_K \frac{d\sigma(k)}{|\rho(gk)w|^\beta} &\leq \int_K \frac{d\sigma(k)}{|\pi_i(\rho(gk)w)|_i^\beta} \\
&= \int_K \frac{d\sigma(k)}{|\rho_i(gk)\pi_i(w)|_i^\beta} \leq \tau_{\beta m(i)}(g) = \frac{1}{|\pi_i(w)|_i^\beta} \\
&= \tau_{\beta m(i)}(g)\frac{1}{|w|^\beta} \leq \max_{m \in I(\rho)} \{\tau_{\beta n}(g)\}\frac{1}{|w|^\beta}.
\end{aligned}$$

Thus it is enough to prove the proposition for representations ψ_m. □

Let $P \in \mathcal{P}_m, P \neq 0$. We consider P as a polynomial on $\mathbb{C}^2$ and decompose P into the product of m linear forms

$$P = \ell_1 \cdot \ldots \cdot \ell_m, \ell_i(z_1, z_2) = a_i z_1 + b_i z_2, a_i, b_i z_1, z_2 \in \mathbb{C}.$$

There is a natural K-invariant norm on the space of linear forms on $\mathbb{C}^2$:

$$\|\ell\|^2 = |a|^2 + |b|^2,\ \ell(z_1, z_2) = az_1 + bz_2.$$

Now we define a quasinorm on $\mathcal{P}_m$ by the equality

$$|P| = \|\ell_1\| \cdot \ldots \cdot \|\ell_m\|. \tag{6.50}$$

This definition is correct because the factorization (6.50) is unique up to the order of factors and the multiplication of ℓ_i, $1 \leq i \leq n$, by constants. We denote by $\tilde{\psi}_1$ the extension of ψ_1 to the space of linear forms on G. It is isomorphic to the standard representation of G on $\mathbb{C}^2$. Then using Lemma 6.7 and the generalized Hölder inequality,

we get that

$$\int \frac{d\sigma(k)}{|\psi_m(gk)P|^\beta} = \int_K \frac{d\sigma(k)}{\prod_{i=1}^m \|\tilde\psi_1(gk)\ell_i\|^\beta} \leq \prod_{i=1}^m \left(\int_K \frac{d\sigma(k)}{\|\tilde\psi_1(gk)\ell_i\|^{\beta m}}\right)^{1/m}$$

$$\leq \prod_{i=1}^m \left(\frac{\tau_{\beta m}(g)}{\|\ell_i\|^{\beta m}}\right)^{1/m} = \frac{\tau_{\beta m}(g)}{|P|^\beta}. \quad (6.51)$$

Since $I(\psi_m) = \{m\}$, (6.51) implies (6.48) for $\rho = \psi_m$.

We remember (see (6.14) and (6.15) from section 6.2 that $\tau_\mu(g) < 1$ and $\tau_\eta(g) < \tau_\lambda(g)$ for any $g \notin K$, $0 < \mu < 2$, $\lambda \geq 2$ and $0 < \eta < \lambda$. Using this, we deduce from Proposition 6.8 the following corollary.

Corollary 6.9. *Let ρ be a representation of $G = SL(2,\mathbb{R})$ in a finite dimensional space W, and let m be the largest number in $I(\rho)$. Then there exists a K-invariant quasinorm $|\cdot| = |\cdot|_\rho$ on W such that*

(i) if $\beta > 0$ and $\beta m \geq 2$ then for any $w \in W$, $w \neq 0$, and $g \in G$,

$$\int_K \frac{d\sigma(k)}{|\rho(gk)w|^\beta} \leq \tau_{\beta m}(g)\frac{1}{|w|^\beta};$$

(ii) if $\beta > 0$ and $\beta m < 2$ then for any $w \in W, w \neq 0$, and $g \in G, g \notin K$,

$$\int_K \frac{d\sigma(k)}{|\rho(gk)w|^\beta} < \frac{1}{|w|^\beta}.$$

6.4. **Functions α_i on the space of lattices and estimates for $A_h\alpha_i$.** For a lattice Δ in R^n recall the notations $d(L)$ and $\alpha_l(L)$ in section 5 for arithmetic subspaces of R^n which satisfy the basic inequalities (5.1) of Lemma 5.1 for two Δ-rational subspaces L and M. Choosing quasinorms $\|\cdot\|_l$ on $\wedge^l\mathbb{R}^n$ we consider the functions $\alpha_l(L)$ and $\alpha(L)$, see (5.2) and (5.3), on the space of lattices.

Let ρ be a representation of $G = SL(2,\mathbb{R})$ in $\mathbb{R}^n$, and let $\wedge^i\rho, 1 \leq i \leq n$, denote the i-th exterior product of ρ. We assume that the quasinorms $|\cdot|_i$ are K-invariant or, more precisely, $(\wedge^i\rho)(K)$-invariant.

We fix $g \in G, g \notin K$. Since

$$\sup\left\{\frac{|(\wedge^i\rho)(h)v|_i}{|v|_i} : h \in A, v \in \wedge^i\mathbb{R}^n, v \neq 0\right\}$$
$$= \sup\{|(\wedge^i\rho)(h)v|_i : h \in A, v \in \wedge^i\mathbb{R}^n, |v|_i = 1\}$$

is finite for every $i, 1 \leq i \leq n$, and every compact subset $A \subset G$, one can find $Q > 1$ such that for any $i, 1 \leq i \leq n$, and $v \in \wedge^i\mathbb{R}^n, v \neq 0$,

$$Q^{-1} < \frac{|(\wedge^i \rho)(h)v|_i}{|v|_i} < Q \quad \text{if } y \in G \text{ and } \|y\| \leq \|g\|, \tag{6.52}$$

where, as in section 6.2, $\|h\| = \|h^{-1}\|$ denotes the norm of the linear transformation $h \in G = SL(2, \mathbb{R})$ with respect to the standard Euclidean norm on $\mathbb{R}^2$. From (6.52) and the definition of $d_\Delta(L)$ we get that, for any lattice Δ in $\mathbb{R}^n$ and any Δ-rational subspace L,

$$Q^{-1} < \frac{d_{y\Delta}(yL)}{d_\Delta(L)} < Q \quad \text{if } y \in G \text{ and } \|y\| \leq \|g\|. \tag{6.53}$$

Hence for any $i, 0 \leq i \leq n$,

$$\alpha_i(y\Delta) < Q\alpha_i(\Delta) \quad \text{if } y \in G \text{ and } \|y\| \leq \|g\|. \tag{6.54}$$

Let $\beta > 0$. We define functions $F_{i,\beta}$, $1 \leq i \leq n$, on $\wedge^i \mathbb{R}^n - \{0\}$ by

$$F_{i,\beta}(w) = \int_K \frac{|w|_i}{|(\wedge^i \rho)(gk)w|_i} \, d\sigma(k), w \in \wedge^i \mathbb{R}^n, w \neq 0.$$

It is clear that the functions F_i are continuous and that $F_{i,\beta}(\lambda w) = F_{i,\beta}(w), \lambda \in \mathbb{R}, \lambda \neq 0$. Let $c_{0,\beta} \stackrel{\text{def}}{=} 1$ and

$$\begin{aligned} c_{i,\beta} &\stackrel{\text{def}}{=} \sup\{F_{i,\beta}(w) : w \in \wedge^i \mathbb{R}^n, w \neq 0\} \\ &= \sup\{F_{i,\beta}(w) : w \in \wedge^i \mathbb{R}^n, |w|_i = 1\}, 1 \leq i \leq n. \end{aligned} \tag{6.55}$$

We note that $c_{n,\beta} = 1$.

Lemma 6.10. *For any $i, 0 \leq i \leq n$,*

$$A_g \alpha_i^\beta \leq c_{i,\beta} \alpha_i^\beta + C^\beta Q^\beta \max_{0 < j \leq \bar{i}} \sqrt{\alpha_{i-j}^\beta \alpha_{i+j}^\beta}, \tag{6.56}$$

where $\bar{i} = \min\{i, n-i\}$, the constant $C \geq 1$ is from Lemma 5.1, and the operator A_g is defined by (6.7) from 6.2.

Proof. Let Δ be a lattice in $\mathbb{R}^n$. We have to prove that

$$\begin{aligned} \int_K \alpha_i(gk\Delta)^\beta d\sigma(k) &\leq c_{i,\beta}\alpha_i(\Delta)^\beta \\ &+ C^\beta Q^\beta \max_{0<j\leq \bar{i}} \sqrt{\alpha_{i-j}(\Delta)^\beta \alpha_{i+j}(\Delta)^\beta} \end{aligned} \tag{6.57}$$

There exists a Δ-rational subspace L of dimension i such that

$$\frac{1}{d_\Delta(L)} = \alpha_i(\Delta). \tag{6.58}$$

Let us denote the set of Δ-rational subspaces M of dimension i with $d_\Delta(M) < Q^2 d_\Delta(L)$ by Ψ_i. We get from (6.53) that

$$d_{gk\Delta}(gkM) > d_{gk\Delta}(gkL)$$

for a Δ-rational i-dimensional subspace $M \notin \Psi_i$. It follows from this and the definitions of α_i and $c_{i,\beta}$ that

$$\int\limits_K \alpha_i(gk\Delta)^\beta d\sigma(k) \le c_{i,\beta}\alpha_i(\Delta)^\beta \quad \text{if } \Psi_i = \{L\}. \tag{6.59}$$

Assume now that $\Psi_i \neq \{L\}$. Let $M \in \Psi_i, M \neq L$. Then $\dim(M+L) = i+j$, $0 < j \le \bar{i}$. Now by (6.53), (6.58) and Lemma 5.1, for any $k \in K$,

$$\begin{aligned}\alpha_i(gk\Delta) &< Q\alpha_i(\Delta) = \frac{Q}{d_\Delta(L)} \le \frac{Q}{\sqrt{d_\Delta(L)d_\Delta(M)}}\\ &\le \frac{CQ}{\sqrt{d_\Delta(L\cap M)d_\Delta(L+M)}}\\ &\le CQ\sqrt{\alpha_{i-j}(\Delta)\alpha_{i+j}(\Delta)}.\end{aligned}$$

Hence, if $\Psi_i \neq \{L\}$,

$$\int\limits_K \alpha_i(gk\Delta)^\beta d\sigma(k) \le C^\beta Q^\beta \max_{0<j\le\bar{i}} \sqrt{\alpha_{-j}(\Delta)^\beta \alpha_{i+j}(\Delta)^\beta}. \tag{6.60}$$

Combining (6.59) and (6.60), we get (6.57). □

Theorem 6.11. *Let $d \in \mathbb{N}^+$, and let ρ_d denote the direct sum of d copies of the standard 2-dimensional representation of $G = SL(2,\mathbb{R})$. Let β be a positive number such that $\beta d > 2$. Then there is a constant R, depending only on β and the choice of the K-invariant quasinorms $|\cdot|_i$ involved in the definition of α_i, such that for $h \in G$ and a lattice Δ in $\mathbb{R}^{2d}$*

$$(A_h\alpha^\beta)(\Delta) = \int\limits_K \alpha(hk\Delta)^\beta d\sigma(k) \le R\alpha(\Delta)^\beta \|h\|^{\beta d - 2}.$$

Proof. As in section 6.3, for a finite dimensional representation ρ of G, we define

$$I(\rho) = \{m \in \mathbb{N}^+ : \psi_m \text{ is isomorphic to a subrepresentation of } \rho\},$$

where ψ_m denotes the regular representation of G in the space of real homogeneous polynomials of degree m. Let m_i be the largest number in $I(\wedge^i\rho_d)$, $1 \le i \le 2d$. It is well known that

$$-\, m_i = \bar{i} \overset{\text{def}}{=} \min\{i, 2d-i\}. \tag{6.61}$$

We fix $g \in G, g \notin K$. It follows from (6.61) and from the Corollary of Proposition 6.8 that we can choose quasi-norms $|\cdot|_i$ on $\wedge^i \mathbb{R}^{2d}$ in such a way that for $w \in \wedge^i \mathbb{R}^{2d}, w \neq 0$,

$$\int_K \frac{|w|_i^\beta}{|(\wedge^i \rho_d)(g) w|_i^\beta} \, d\sigma(k) \leq \tau_{\beta \bar{i}}(g), \quad \text{if } \beta \bar{i} \geq 2$$

and

$$\int_K \frac{|w|_i^\beta}{|(\wedge^i \rho_d)(g) w|_i^\beta} < 1 \quad \text{if } \beta \bar{i} < 2.$$

Hence

$$c_{i,\beta} \leq \tau_{\beta \bar{i}}(g) \quad \text{if } \beta \bar{1} \geq 2, \text{ and} \tag{6.62}$$

$$c_{i,\beta} \leq 1 \quad \text{if } \beta \bar{i} < 2, \tag{6.63}$$

where $c_{i,\beta}, 1 \leq i \leq 2d$, is defined by (6.55) and $c_{0,\beta} = 1$. As a remark, we notice that $c_{i,\beta} = \tau_{\beta \bar{i}}(g)$ if $\beta \bar{i} \geq 2$). □

According to Lemma 6.10, the functions α_i^β, $0 \leq i \leq 2d$, satisfy the following system of inequalities

$$A_g \alpha_i^\beta \leq c_{i,\beta} \alpha_i^\beta + C^\beta Q^\beta \max_{0 < j \leq \bar{i}} \sqrt{\alpha_{i-j}^\beta \alpha_{i+j}^\beta}. \tag{6.64}$$

Let

$$\lambda_i \stackrel{\text{def}}{=} \max\{2, \beta \bar{i}\}, \quad 0 \leq i \leq 2d. \tag{6.65}$$

Since $\tau_2(g) = 1$ (see (6.13) from section 6.2), it follows from (6.62)-(6.65) that

$$A_g \alpha_i^\beta \leq \tau_{\lambda_i}(g) \alpha_i^\beta + C^\beta Q^\beta \max_{0 < j \leq \bar{i}} \sqrt{\alpha_{i-j}^\beta \alpha_{i+j}^\beta}, \quad 0 \leq i \leq 2d. \tag{6.66}$$

Now we fix a lattice Δ in $\mathbb{R}^{2d}$ and define functions f_i, $0 \leq i \leq 2d$, on G by

$$f_i(h) = \alpha_i(h\Delta)^\beta, \quad h \in G.$$

Then it follows from (6.66) that

$$A_g f_i \leq \tau_{\lambda_i}(g) f_i + C^\beta Q^\beta \max_{0 < j \leq \bar{i}} \sqrt{f_{i-j} f_{i+j}}, \quad 0 \leq i \leq 2d.$$

On the other hand, in view of (6.54),

$$f_i(yh) \leq Q^\beta f_i(h) \quad \text{if } 0 \leq i \leq 2d, h \in G, y \in G \text{ and } \|y\| \leq \|g\|.$$

Since $\beta d > 2$, we have that $\beta d - 2 = \lambda_d > \lambda_i$ for any $i \neq d$. Now we can apply Proposition 6.8 (c) and get that

$$\begin{aligned}
&(A_h \alpha^\beta)(\Delta) < (A_h \sum_{0 \le i \le 2d} \alpha_i^\beta)(\Delta) = (A_h \sum_{0 \le i \le 2d} f_i)(1) \\
&\ll (\sum_{0 \le i \le 2d} f_i(1)) \|h\|^{\lambda_d - 2} \\
&= (\sum_{0 \le i \le 2d} \alpha_i(\Delta)^\beta) \|h\|^{\lambda_d - 2} \le 2d \alpha(\Delta)^\beta \|h\|^{\beta d - 2}. \qquad (6.67)
\end{aligned}$$

The inequality (6.67) proves the theorem for our specific choice of the quasinorms $|\cdot|_i$. Now it remains to notice that any two quasinorms on $\wedge^i \mathbb{R}^n$ are equivalent.

7. Conclusion of the Proofs

We shall apply Theorem 6.11 combined with Lemma 6.1 as follows.

Corollary 7.1. *Let $|t_0| > 2$, $r > r_Q$, $I = [t_0 - 2, t_0 + 2]$ and $\beta d > 2$ and $\widehat{g}_I \stackrel{\text{def}}{=} \max\{|\widehat{g}_w(t)| : t \in I\}$. With the notations of Lemma 6.1 we have*

$$\sup_{v \in \mathbb{R}^d} \int_I \Big|\theta_v(t) \widehat{g}_w(t)\Big| dt \quad \ll_d \quad \widehat{g}_I C_Q \gamma_{I,\beta}(r) r^{d-2}, \qquad (7.1)$$

where $C_Q \stackrel{\text{def}}{=} r_Q^2 |\det Q|^{-1/4 - \beta/2}$.

Proof. Note that $\alpha_d(\Lambda) \le \alpha(\Lambda)$ holds for any lattice Λ. Since $d_r u_t \Lambda_{Q,s} = d_{r_Q} u_s \Lambda_Q$, (compare (6.1)), is self-dual, (5.26) applies and we obtain $\alpha(d_r u_t \Lambda_{Q,s}) \asymp_d \alpha_d(d_r u_t \Lambda_{Q,s})$. Now choosing $h = d_{r_0}$, $r_0 = r/r_Q$, $r_Q = q^{1/2}$ and the lattices $\Delta = \Lambda_{Q,s_j}$ defined in in Theorem 6.11 we arrive at the bound

$$\max_{j \in J} \int_{-\pi}^{\pi} \alpha_d(d_{r_0} k_\theta \Lambda_{Q,s_j})^\beta \frac{d\theta}{2\pi} \ll_d \alpha(\Lambda_{Q,s_j})^\beta \|d_{r_0}\|^{\beta d - 2} \ll_d r^{\beta d - 2} \Big(r_Q^2 |\det Q|^{-\beta/2}\Big),$$

using $\|d_{r_0}\| = r_0$ and by (5.34),

$$\alpha(\Lambda_{Q,s}) \ll_d \alpha_d(\Lambda_{Q,s}) \ll_d |\text{det} Q|^{-1/2} r_Q^d. \qquad (7.2)$$

In view of Lemma 6.1 this concludes the proof of (7.1). □

In the intervals $\{r^{-1} \le |t| \le 1\}$ the integrand $\widehat{g}_w(t), w < 1$ in (6.2) is bounded from above by, (see (4.3)),

$$|\widehat{g}_w(t)| \ll \min\{|b - a|, 1/|t|\} \qquad (7.3)$$

and thus is of size $b - a$ for $|t| \le 1/(b - a)$. It changes rapidly if $|b - a| > 1$ grows with r. In order to adapt the averaging result, Lemma 7.1, to this case we need the following Lemma.

Lemma 7.2. *Recall $r_Q = q^{1/2}$. For $r \geq r_Q$, $b - a > r_Q$, $\beta d > 2$ and $0 < w < 1$ we have with $I \stackrel{\text{def}}{=} [1/r, 1/r_Q]$*

$$I_1(v) \stackrel{\text{def}}{=} \int_{r^{-1}}^{r_Q^{-1}} |\theta_v(t)\widehat{g}_w(t)|\, dt \ll C_Q \gamma_{I,\beta}(r) r^{d-2}, \tag{7.4}$$

with C_Q as defined in Corollary 7.1 .

Proof. If $|b - a| > r_Q^{-1}$, using (7.4) and Lemma (4.4) we have with $d\,\beta > 2$, Λ_Q as defined in (5.18) and (6.3) with $t_0 = 0$

$$I_1(v) \ll_d \gamma_{I,\beta}(r)\, r^{(1/2-\beta)d} |\det Q|^{-1/4} J, \tag{7.5}$$

where

$$J = \int_{r^{-1}}^{r_Q^{-1}} (\alpha_d(d_r\, u_t\, \Lambda_Q))^{\beta}\, |\widehat{g}_w(t)|\, dt \leq \sum_{j=j_0}^{\rho} I_j, \tag{7.6}$$

with

$$I_j \stackrel{\text{def}}{=} \int_{j^{-1}}^{(j-1)^{-1}} \alpha_d(d_r\, u_t\, \Lambda_Q)^{\beta}\, |\widehat{g}_w(t)|\, dt, \quad j = j_0, j_0+1, \ldots, \rho \stackrel{\text{def}}{=} \lceil r \rceil + 1. \tag{7.7}$$

and $j_0 \stackrel{\text{def}}{=} \lceil r_Q \rceil$.

Changing variables via $t = v\, j^{-2}$ and $v = s + j$ in I_j and using the group properties of d_r and u_t, we have with $|\widehat{g}_w(t)| \ll |t|^{-1}$ for $|t| \leq r_Q^{-1}$ and $I_j \leq I_j^*$

$$\begin{aligned} I_j^* &\stackrel{\text{def}}{=} \int_{j^{-1}}^{(j-1)^{-1}} (\alpha_d(d_r\, u_t\, \Lambda_Q))^{\beta}\, \frac{dt}{t} \\ &= \int_{j}^{j^2(j-1)^{-1}} (\alpha_d(d_r\, u_{vj^{-2}}\, \Lambda_Q))^{\beta}\, \frac{dv}{v} \\ &\leq \int_{j}^{j+2} (\alpha_d(d_r\, u_{vj^{-2}}\, \Lambda_Q))^{\beta}\, \frac{dv}{v} \\ &= \int_{0}^{2} (\alpha_d(d_r\, u_{sj^{-2}}\, u_{j^{-1}}\, \Lambda_Q))^{\beta}\, \frac{ds}{s+j}\,. \end{aligned} \tag{7.8}$$

By (5.27),

$$d_r\, u_{sj^{-2}} = d_{rj^{-1}}\, d_j\, u_{sj^{-2}} = d_{rj^{-1}}\, u_s\, d_j. \tag{7.9}$$

According to (7.8) and (7.9),

$$I_j^* \ll \frac{1}{j} \int_0^2 (\alpha_d(d_{rj^{-1}} u_t \Lambda_j))^{\beta}\, dt, \tag{7.10}$$

where $\Lambda_j \stackrel{\text{def}}{=} d_j\, u_{j^{-1}} \Lambda_Q$. By Lemma 5.6, (5.34) we have for $j \geq j_0$ and

$$\alpha_d(\Lambda_j) \ll_d |\det Q|^{-1/2}\, r_Q^d. \tag{7.11}$$

By Lemma 5.9 (with $t_0 = 0$), (6.2), we have

$$\alpha_d(d_{rj^{-1}} u_t \Lambda_j) \ \ll_d \ \alpha_d(d_{rj^{-1}}\, k_\theta\, \Lambda_j), \tag{7.12}$$

for $|t| \le 1$, $r \ge 1$, where k_θ is defined in (6.2). Recall that $\alpha_d(\Lambda) \le \alpha_(\Lambda)$ for any $2d$-dimensional lattice Λ. Choosing $r = j$ and $t = j^{-1}$, note that $\Lambda_t = \Lambda_j$ and (5.26) yields

$$\alpha_d(\Lambda_j) \asymp_d \alpha(\Lambda_j). \tag{7.13}$$

Thus Theorem 6.11, with $\Lambda = \Lambda_j$ and r replaced by rj^{-1}, together with (7.13) yields

$$\int_0^{2\pi} \big(\alpha_d(d_{rj^{-1}}\, k_\theta\, \Lambda_j)\big)^\beta\, d\theta \ll_d \|\, d_{rj^{-1}}\|^{\beta d-2} \alpha(\Lambda_j)^\beta \ll_d \|\, d_{rj^{-1}}\|^{\beta d-2} \alpha_d(\Lambda_j)^\beta. \tag{7.14}$$

Using (5.43), (7.12) and (7.14) we have

$$\begin{aligned} \int_0^2 \big(\alpha_d(d_{rj^{-1}} u_t \Lambda_j)\big)^\beta\, dt \ &\ll_d \ \int_0^{c^*} \big(\alpha_d(d_{rj^{-1}}\, k_\theta\, \Lambda_j)\big)^\beta\, \frac{d\theta}{\cos^2\theta} \\ &\ll \ \int_0^{2\pi} \big(\alpha_d(d_{rj^{-1}}\, k_\theta\, \Lambda_j)\big)^\beta\, d\theta \\ &\ll_d \ \|\, d_{rj^{-1}}\|^{\beta d-2} \alpha_d(\Lambda_j)^\beta, \end{aligned} \tag{7.15}$$

if $d \ge 5$, $\beta > 2/d$. It is clear that $\|\, d_{rj^{-1}}\| = r\, j^{-1}$. Therefore, according to (7.10) and (7.15),

$$I_j^* \ll_d \frac{1}{j}\, (rj^{-1})^{\beta d-2}\, (\alpha_d(\Lambda_j))^\beta \ll_d \frac{1}{j}\, (rj^{-1})^{\beta d-2} |\det Q|^{-\beta/2} r_Q^{\beta\, d}. \tag{7.16}$$

By (7.6), (7.11) and (7.16), we obtain, for $d \ge 5$, $\beta > 2/d$, $w \le 1$,

$$J \ll_d \sum_{j=j_0}^{\rho} \frac{1}{j}\, (rj^{-1})^{\beta d-2}\, |\det Q|^{-\beta/2} r_Q^{\beta d} \ll_d r^{\beta d-2} |\det Q|^{-\beta/2} r_Q^2. \tag{7.17}$$

Collecting the bounds (7.5)–(7.6) and (7.17), we get the bound for $I_1(v)$. □

7.1. **Proof of Theorem 2.1.** We now collect all error bounds to prove Theorem 2.1 using the constants C_Q and $r_Q = q^{1/2}$ introduced in Corollary 7.1.

Estimate of $I_1 \stackrel{\text{def}}{=} I_{1,\pm}$ in (4.4) and (4.37).

With $K_0 \stackrel{\text{def}}{=} [r^{-1}, 1]$ and $K_j \stackrel{\text{def}}{=} [j, j+1]$, $j \ge 1$, we have

$$I_1 \le I_{1,0} + \sum_{j=1}^{\infty} I_{1,j}\,, \tag{7.18}$$

where

$$I_{1,j} \ \stackrel{\text{def}}{=} \ \int_{\mathbb{R}^d} dv \int_{K_j} dt \Big|\theta_v(t)\, \widehat{g}_w(t)\Big|, \qquad j \ge 0\,. \tag{7.19}$$

Write

$$(b-a)^* \stackrel{\text{def}}{=} \frac{1}{2} \min(r_Q, b-a). \tag{7.20}$$

Estimate of $I_{1,0}$. Using (4.38) we consider the case $b-a \le 1$ first. Here we use Corollary 7.1 to bound the integral over K_0 using $\widehat{g}_{K_0} \ll b-a = (b-a)^*$. For the case $b-a>1$ we use Lemma 7.2 for $t \in K_0, |t| \le r_Q^{-1}$ and Corollary 7.1 for the other t in K_0 together with $\widehat{g}_{[r_Q^{-1},1]} \ll (b-a)^*$. The result is

$$I_{1,0} \ll_d (b-a)^* C_Q (\log \varepsilon^{-1})^d \gamma_{K_0,\beta}(r)\, r^{d-2}. \tag{7.21}$$

Estimate of $I_{1,j}$, $j \ge 1$.
Using Corollary 7.1 we get

$$I_{1,j} \ll_d \widehat{g}_{K_j} (\log \varepsilon^{-1})^d C_Q \gamma_{K_j,\beta}(r)\, r^{d-2} \tag{7.22}$$

Choice of parameters:
For fixed $r > q$ and $|a|+|b| < c_0 r^2, \quad 0 < b-a < c_0 r^2$ we may choose $w < (b-a)/4$ and prescribe limits for $T > 2$ and $1 \ge T_- > 1/r$:

$$r^{-1}(\log r)^2 \ll_d \varepsilon, \quad 0 < w < (b-a)^*/4, \quad T \gg_d w^{-1} h(T_-), \tag{7.23}$$

where c_0 is defined in Lemma 3.3 and $h(T_-) \overset{\text{def}}{=} c^{-1}\Big(\log(T_- q^{-1/2})^{\varsigma}(b-a)^*\Big)^2$.

Recall that

$$|\widehat{g}_w(t)| \le \min\{|b-a|, |t|^{-1}\} \exp\{-c|tw|^{1/2}\}. \tag{7.24}$$

Thus for the choices of T and w in (7.23) we have for sufficiently small w,

$$\sum_{j=T}^{\infty} \widehat{g}_{K_j} \ll \exp\{-c|Tw|^{1/2}\}. \tag{7.25}$$

Using Corollary 7.1 and Lemma 5.8, (5.34), we conclude with $\bar{c}_Q \overset{\text{def}}{=} \Big(|\det Q|^{-1/2} r_Q^d\Big)^{1/2-\beta}$, $\sum_{j \ge T} I_{1,j} \ll_d (\log \varepsilon^{-1})^d r^{d-2} C_Q \bar{c}_Q \exp\{-c|Tw|^{1/2}\}$. Furthermore, for $b-a > r_Q$ and $j \ge 1$ we have $|\widehat{g}_{K_j}| \ll j^{-1}\exp\{-c|jw|^{1/2}\}$ whereas for $b-a \le r_Q$ and $1 \le j \le w^{-1}$ we obtain $|\widehat{g}_{K_j}| \ll \min\{b-a, j^{-1}\}$. Thus with $(b-a)^*$ defined in (7.20) we get

$$\sum_{j=1}^{T} \widehat{g}_{K_j} \ll \log \frac{(b-a)^*}{w}. \tag{7.26}$$

Hence, in view of (7.18), (7.19), (7.21) and (7.22) we obtain (using the notation in (6.1))

$$\begin{aligned} I_1 \ll_d\ & (\log \varepsilon^{-1})^d r^{d-2} C_Q \Big((b-a)^* \gamma_{K_0,\beta}(r) + \gamma_{[1,T],\beta}(r) \log \frac{(b-a)^*}{w} \\ & + \frac{\bar{c}_Q}{(b-a)^*} \exp\{-c(Tw)^{1/2}\}\Big). \end{aligned} \tag{7.27}$$

Split $K_0 = K_{00} \cup K_{01}$, where $K_{00} \overset{\text{def}}{=} [r^{-1}, T_-]$ and $K_{01} \overset{\text{def}}{=} (T_-, 1]$. Then (5.35) yields

$$\gamma_{K_{00},\beta}(r) \ll_d \big(|\det Q|^{1/2} T_-^d\big)^{1/2-\beta} \ll T_-^{\varsigma} |\det Q|^{1/2(1/2-\beta)}. \tag{7.28}$$

Write $c_Q \overset{\text{def}}{=} |\det Q|^{1/4-\beta/2} \ge 1$ and note that $\gamma_{K_{01},\beta}(r) \ll_d \gamma_{[T_-,T],\beta}(r)$

Thus, we finally get

$$\begin{aligned} I_1 \quad \ll_d \quad & (\log \varepsilon^{-1})^d r^{d-2} C_Q \Big(\chi_{w,b-a} \gamma_{[T_-,T],\beta}(r) + (b-a)^* c_Q T_-^{\varsigma} + \\ & + \bar{c}_Q \exp\{-c(Tw)^{1/2}\} \Big). \end{aligned} \tag{7.29}$$

Together with the inequalities (3.3) and (4.33), (4.35), (4.38) we conclude, for $T > w^{-1} h(T_-)$,

$$\begin{aligned} & |\mathrm{vol}_{\mathbb{Z}} H(r) - \mathrm{vol}\, H(r)| \qquad (7.30) \\ \ll_d \quad & (\log \varepsilon^{-1})^d r^{d-2} C_Q (b-a)^* \Big(\log \frac{(b-a)^*}{w} \gamma_{[T_-,T],\beta}(r) + c_Q T_-^{\varsigma} + \bar{c}_Q \exp\{-c(Tw)^{1/2}\} \Big) \\ & + ((b-a)\varepsilon + w) |\det Q|^{-1/2} r^{d-2} + (\log \varepsilon^{-1})^d |\det Q|^{-1/2} r^{d/2} \log(1 + |b-a| r^{-1}). \end{aligned}$$

Since by (3.8), $\mathrm{vol}\, H(r) \gg_d (b-a)|\det Q|^{-1/2} r^{d-2}$ and since $\bar{c}_Q \exp\{-c(Tw)^{1/2}\} < c_Q T_-^{\varsigma}(b-a)^*$ for T chosen in (7.23), we conclude for the relative lattice point deficiency in Theorem 2.1,

$$\begin{aligned} \Delta_r & \overset{\text{def}}{=} \Big| \frac{\mathrm{vol}_{\mathbb{Z}} H(r)}{\mathrm{vol}\, H(r)} - 1 \Big| \qquad (7.31) \\ & \ll_d \quad \varepsilon + \frac{w}{b-a} + (\log \varepsilon^{-1})^d \Big(|\det Q|^{1/2} C_Q \frac{\rho^*_{Q,b-a,w}(r)}{b-a} + r^{-d/2} \xi(r, b-a) \Big), \end{aligned}$$

where

$$\begin{aligned} \rho^*_{Q,b-a,w}(r) \quad & \overset{\text{def}}{=} \quad \inf^*_{T_-,T} \Big\{ \gamma_{[T_-,T],\beta}(r) \log \frac{(b-a)^*}{w} + (b-a)^* c_Q T_-^{\varsigma} \Big\}, \\ \xi(r, b-a) \quad & \overset{\text{def}}{=} \quad \frac{r^2 \log(1 + \frac{b-a}{r})}{b-a}, \end{aligned} \tag{7.32}$$

and $\inf^*_{T_-,T}$ denotes the infimum over all $T_- \in [r^{-1}, 1]$ and $T > h(T_-)/w$.

Fixed parameters ε and w:

When $0 < b-a \le r_Q$, we may rewrite (7.31) and (7.32) as follows:

$$\Delta_r \ll_d \varepsilon + \frac{w}{b-a} + (\log \varepsilon^{-1})^d \Big(|\det Q|^{1/2} C_Q \rho_{Q,b-a,w}(r) + r^{-d/2+1} \Big), \tag{7.33}$$

where

$$\rho_{Q,b-a,w}(r) \overset{\text{def}}{=} \inf \Big\{ \frac{\log(w^{-1}(b-a))}{b-a} \gamma_{[T_-,T],\beta}(r) + c_Q T_-^{\varsigma} \,:\, T_- \in [r^{-1}, 1],\, T > h(T_-)/w \Big\}. \tag{7.34}$$

When $r > b-a > r_Q$ we get a similar bound

$$\Delta_r \ll_d \varepsilon + \frac{w}{b-a} + (\log \varepsilon^{-1})^d C_Q |\det Q|^{1/2} \Big(\frac{\rho_{Q,w}(r)}{b-a} + r^{-d/2+1} \Big), \tag{7.35}$$

with

$$\rho_{Q,w}(r) \stackrel{\text{def}}{=} \inf\Big\{(\log w^{-1})\gamma_{[T_-,T],\beta}(r)+c_Q r_Q T_-^{\varsigma}\ :\ T_- \in [r^{-1},1],\ T \gg_d h(T_-)/w\Big\} \quad (7.36)$$

whereas for $c_0 r^2 > b-a \geq r$ we get for $0<w<1$ in view of (7.23)

$$\Delta_r \ll_d \varepsilon + (\log \varepsilon^{-1})^d\Big(C_Q|\det Q|^{1/2}\,\frac{\rho_{Q,w}(r)}{b-a} + r^{-d/2}\,\frac{r^2}{b-a}\,\log(r)\Big). \quad (7.37)$$

Variable parameters ε and w:
Finally, we may choose $w = T_-^{\varsigma}(b-a)^*$ and $\varepsilon = T_-^{\varsigma}$ subject to the restrictions in (7.23). Using $|\det Q| \geq 1$ this results in the following bound:

$$\Delta_r \ll_d |\det Q|^{1/2} C_Q\,\frac{\rho_Q(r,b-a)}{b-a} + r^{-d/2}\xi(r,b-a),$$

with $T_+ \asymp_d h(T_-)T_-^{-\varsigma}$ and

$$\rho_{Q,b-a}(r) \stackrel{\text{def}}{=} \inf\Big\{|\log T_-|^d\Big(\gamma_{[T_-,T_+],\beta}(r)\log\frac{(b-a)^*}{T_-^{\varsigma}}+c_Q\,(b-a)^* T_-^{\varsigma}\Big)\ :\ T_- \in [r^{-1}+r^{-1/\varsigma},1]\Big\}, \quad (7.38)$$

where $\xi(r,b-a) \ll r$ if $b-a<r$ and $\xi(r,b-a) \ll r^2(b-a)^{-1}\log(r)$ if $b-a>r$. Since $\gamma_{[T_-,T_+],\beta}(r) \to 0$ for $r\to\infty$ (and any fixed T_-) when Q is irrational, we conclude that $\rho_Q^*(r,b-a,w) \to 0$ for $r \to \infty$, which proves Theorem 2.1.

Proof of Corollary 2.3. Choose $r = 2\sqrt{b}$. Then $E_{a,b} \subset C_{r/2} \stackrel{\text{def}}{=} \{x \in \mathbb{R}^d\ :\ L_Q x \in [-1,1]^d\}$. Choosing $\varepsilon = 1/16$ in Lemma 3.3 we get $R_{\varepsilon,w,2r} \ll_d |\det Q|^{-1/2}\,w\,r^{d-2}$ for smoothing regions since the boundary of $E_{a,b}$ does not intersect the 3ε-boundary of C_r. Furthermore, in Lemma 3.3 note that with these choices $\operatorname{vol} H(r) = \operatorname{vol} C_r \cap E_{a,b}) \asymp_d |\det Q|^{-1/2} r^d$, see [BG99]. Moreover, by the choice of ε and r all lattice points in $E_{a,b}$ will receive the weight 1. Accordingly we set the terms estimating the intersection with the 3ε-boundary of C_r to zero, i.e. drop the terms $(b-a)\varepsilon$ in (3.19), (4.37), and the summands ε in (7.30) and (7.31). Note that apart from Lemma 3.3 the property that Q is *indefinite* has not been used in all the subsequent arguments(!). Thus (7.30) leads to the following bound using $0<w<1$

$$\begin{aligned} &|\operatorname{vol}_{\mathbb{Z}} H_r - \operatorname{vol} H_r| \\ \ll_d\ & r^{d-2}\Big(|\det Q|^{-1/2}\,w + C_Q\,q^{1/2}\Big(\log(w^{-1})\,\gamma_{[T_-,T_+],\beta}(r) + c_Q T_-^{\varsigma}\Big)\Big) \\ &\qquad + |\det Q|^{-1/2}\,r^{d/2}\log r. \end{aligned} \quad (7.39)$$

Thus, choosing $w<1$ such that $|\det Q|^{-1/2}\,w = q^{1/2}\,C_Q\,c_Q T_-^{\varsigma}$ we get with
$T_+ = B_Q T_-^{-\varsigma} h(T_-)$, $\quad B_Q = |\det Q|^{-(1/2-\beta}q^{-3/2}$ where $h(T_-) \asymp_d \Big(\log(T_-\,q^{-1/2})^{\varsigma} q^{1/2}\Big)^2$
and

$$\rho_Q(r) \stackrel{\text{def}}{=} \inf\Big\{\log(T_-^{-1}q)\,\gamma_{[T_-,T_+],\beta}(r) + c_Q T_-^{\varsigma}\ :\ T_- \in [r^{-1},1]\Big\}, \quad (7.40)$$

$$|\operatorname{vol}_{\mathbb{Z}} H_r - \operatorname{vol} H_r| \ll_d r^{d-2}\rho_Q(r) + |\det Q|^{-1/2}\,r^{d/2}\log r. \quad (7.41)$$

Note that as in the indefinite case $\lim_{r\to\infty}\rho(r) = 0$ if Q is irrational. This proves Corollary 2.3.

Proof of Corollary 2.7. Using Corollary 5.11 we may estimate $\rho_{Q,b-a}(r)$ and Δ_r in (7.38). By (5.48) and (6.1), $\gamma_{[T_-,T_+],\beta}(r) \le |\mathrm{det}Q|^{\mu/2} r^{-\nu} \max\Big(T_-^{-(\kappa+1)}, T_+^{\kappa}\Big)^{\mu}$, where $\mu \stackrel{\mathrm{def}}{=} 1/2-\beta, \nu \stackrel{\mathrm{def}}{=} 2(1-\kappa)\mu, T_+ = h(T_-)T_-^{-\varsigma}$, the evaluation of $\rho_{Q,b-a}(r)$ in (7.38) leads to the minimization of $s \to |\log s|^d \Big(c_{Q,(b-a)^*} r^{-\nu} \max\Big\{s^{-1-\kappa}, s^{-\varsigma\kappa}|\log s|^{2\kappa}\Big\}^{\mu} + c_Q\,(b-a)^*\, s^{\varsigma}\Big)$ for $1 \ge s \ge r^{-1} + r^{-1/\varsigma}$, where $c_{Q,(b-a)^*}$ is a polynomial in $|\log(b-a)^*|$ and $\varsigma = \mu d$, (see Theorem 2.1). For sufficiently small $\kappa > 0$ (depending on d), $s^{-\kappa-1}$ dominates $s^{-\varsigma\kappa}|\log s|^{2\kappa}$. Minimizing the resulting function of s, we obtain for r sufficiently large $\rho_{Q,b-a}(r) \ll_{d,Q} (\log r)^{2d} r^{-\nu_1}((b-a)^*)^{1-\nu_2}$, where $\nu_1 = \nu/((1+\kappa)\mu+\varsigma)$ and $\nu_2 = 1/((1+\kappa)\mu+\varsigma)$. Thus (7.38) implies

$$\Delta_r \ll_{Q,d,\kappa,A} r^{-\nu_1}((b-a)^*)^{-\nu_2} + r^{-d/2}\xi(r, b-a), \tag{7.42}$$

which proves Corollary 2.7 when rewriting $\nu_1 = 2(1-\kappa)/(d+1+\kappa)$ and $\nu_2 = 1/(d+1+\kappa)/\mu$.

Proof of Corollary 1.5. It suffices to prove that $\mathrm{vol}_{\mathbb{Z}^d}(C_r \cap E_{a,b}) > 0$ for any $a, b \in [-c_0 r^2, c_0 r^2]$. and $c_{d,\kappa,A,Q} r^{-\nu_0} < b-a$ with a sufficiently large constant $c_{d,\kappa,A,Q}$. Assuming that $b-a > r^{-1}$ and r is sufficiently large, (3.8) shows that $\mathrm{vol}(C_r \cap E_{a,b}) \gg_d |\det Q|^{-1/2}(b-a)r^{d-2} \ge 2$ by (3.8). By Corollary 2.7, denoting the implied constant in (2.14) by $a_{d,D,\kappa,Q}$, say, we have to show that for large r $a_{d,\kappa,A,Q}(r^{-\nu_1}((b-a)^*)^{-\nu_2} + r^{-d/2}\xi(r, b-a)) < 1/2$. For $b-a < r/2$ we obtain $a_{d,\kappa,A,Q} r^{-d/2}\xi(r, b-a) < 1/4$. Hence we should choose $b-a$ large enough, such that $a_{d,D,\kappa,Q} r^{-\nu_1}((b-a)^*)^{-\nu_2} < 1/4$ holds. This holds (and is compatible with the asumption $b-a > r^{-1}$ above), once we require that $b-a > (a_{d,\kappa,A,Q} r^{-\nu_1})^{1/\nu_2} \gg_{d,\kappa,A,Q} r^{-\nu_1/\nu_2}$. Thus the result of Corollary 1.5 follows with an exponent $\nu_0 = \nu_1/\nu_2 = (1-\kappa)/(1-(4+\delta)/d)$.

Proof of Theorem 1.4. For $d \ge 5$ and $q_0 \ge 1$ and $\beta = 2/d + \delta/d$, $\varsigma = d(1/2-\beta) = d/2 - 2 - \delta$ for an arbitrary small $\delta > 0$ and $\varepsilon = \varepsilon_0 < 1/9$, such that $a(d)\varepsilon_0 = 1/16$ we shall choose the other parameters $r = R_Q$, $T = T_Q$, $T_- = T_{Q-}$, and the interval $[a, b] = [-1, 1]$ and $w = w_0$ such that $a(d)\dfrac{w}{2} = 1/16$ (subject to restrictions). Then we split the error terms of Theorem 2.1, see (7.31), as follows

$$\begin{aligned}
\Big|\frac{\mathrm{vol}_{\mathbb{Z}} H(r)}{\mathrm{vol}\, H(r)} - 1\Big| \quad &\le \quad 1/16 + 1/16 + J_1 + J_2 + J_3 + J_4, \quad \text{say, where}\\
J_1 \quad &\stackrel{\mathrm{def}}{=} \quad E_Q\, c_Q\, T_{Q-}^{\varsigma}, \quad J_2 \stackrel{\mathrm{def}}{=} E_Q\, \bar c_Q \exp\{-c(T_Q\, w_0)^{1/2}\}), \quad J_3 \stackrel{\mathrm{def}}{=} E_Q\, R_Q^{-d/2+1},\\
J_4 \quad &\stackrel{\mathrm{def}}{=} \quad E_Q\, \gamma_{[T_{Q-}, T_Q],\beta}, \quad E_Q \stackrel{\mathrm{def}}{=} a(d)\,(\log \varepsilon_0^{-1})^d |\det Q|^{1/4} \Big(|\det Q|^{-\beta/2} q\Big).
\end{aligned}$$

In the following $c(d)$ denotes a generic constant depending on d only which may change from one occurrence to the next. Choose

$$r = R_Q \stackrel{\mathrm{def}}{=} c(d) |\det Q|^{1/2} q^{1/(1/2-\beta)} (\log q)^{(d-1)/2}. \tag{7.43}$$

Then $J_3 \le 1/8$, since $R_Q \gg_d (|\det Q|^{1/4-\beta/2} q)^{2/(d-2)}$. Furthermore, choosing $T_- = T_{Q-} = c(d)|\det Q|^{-1/d} q^{-1/\varsigma}$ we obtain $J_1 \le 1/8$ since $(|\det Q|^{1/2} C_Q c_Q)^{1/\varsigma} = |\det Q|^{1/d} q^{1/\varsigma}$. Similarly choosing $T = T_Q = c(d) w_0^{-1} \log q$ we obtain $J_2 \le 1/8$, since $|\det Q|^{1/2} C_Q \bar{c}_Q = q^{1+\varsigma/2}$.

With these choices we shall distinguish now the following

Case 1. Assume that for the above choices, $J_4 \le 1/8$, that is

$$a(d)\,(\log \varepsilon_0^{-1})^d\,|\det Q|^{1/2}\, C_Q\, \gamma_{[T_{Q-},T_Q],\beta}(R_Q) \le 1/8. \tag{7.44}$$

Then (3.8) implies $\mathrm{vol}_{\mathbb{Z}}\, H(r) > \frac{3}{8} \mathrm{vol}\, H(r) \gg_d R_Q^{d-2}/|\det Q|^{1/2} > 2$ provided that $R_Q \gg_d |\det Q|^{1/(2d-4)}$ which holds for our choice of R_Q. Hence there exist a nontrivial solution $m \in \mathbb{Z}^d$ of $|Q[m]| < 1$ in the support of the smooth box which is contained in the set of x with $\|Q_+^{1/2} x\|_\infty \le R_Q(1+3\varepsilon_0)$. Thus $Q_+[m] \ll_d |\det Q|$, which proves Theorem 1.4 in this case.

Case 2. Assume that for the choices above $J_4 > 1/8$, that is

$$\gamma_{[T_{Q-},T_Q],\beta}(R_Q)^{-1} = \inf_{t\in[T_{Q-},T_Q]} \Big(R_Q^{-d}\alpha_d(\Lambda_t)\Big)^{-(1/2-\beta)} < 8\,a(d)\,(\log \varepsilon_0^{-1})^d\,|\det Q|^{1/2}\, C_Q. \tag{7.45}$$

Thus there exists a $t = t_0 \in [T_{Q-}, T_Q]$ such that the reciprocal α_d-characteristic satisfies, see (2.4) and (2.5),

$$R_Q^d \inf\Big\{ \mathrm{vol}(L_d/L_d \cap \Lambda_{t_0}) \,:\, L_d \subset \mathbb{R}^{2d}, \dim L_d = d\Big\} < D_Q, \tag{7.46}$$

where $D_Q \overset{\mathrm{def}}{=} (8\,a(d)\,(\log \varepsilon_0^{-1})^d\,|\det Q|^{1/2}\, C_Q)^{1/(1/2-\beta)}$. Here the infimum is taken over all linear subpaces L_d of dimension d and $\Lambda_{t_0} = V_{Q,t_0}(Z^d \times \mathbb{Z}^d)$ is the $2d$-dimensional lattice in $\mathbb{R}^{2d}$ induced by the linear map

$$V_{Q,t_0}(m,n) \overset{\mathrm{def}}{=} \Big(R_Q\,(Q_+^{-1/2} m - t_0\, S\, Q_+^{1/2} n), R_Q^{-1}\, Q_+^{1/2}\, n\Big), \tag{7.47}$$

such that $Q = Q_+^{1/2} S Q_+^{1/2}$; compare (5.16). Let L_d denote the subspace in (7.46) such that $R_Q^d\, \mathrm{vol}(L_d/L_d \cap \Lambda_{t_0}) \le D_Q$ and let $\Lambda \overset{\mathrm{def}}{=} R_Q(L_d \cap \Lambda_{t_0})$ denote a corresponding rescaled d-dimensional sublattice of L_d. Then

$$\det \Lambda \le \frac{R_Q^d}{\alpha_d(\Lambda_{t_0})} < D_Q. \tag{7.48}$$

Any element N of the lattice Λ may be written as

$$(N_1, N_2) \overset{\mathrm{def}}{=} \Big(R_Q^2\,(Q_+^{-1/2} m - t_0\, S\, Q_+^{1/2} n), Q_+^{1/2}\, n\Big), \quad (m,n) \in (\mathbb{Z}^d \times \mathbb{Z}^d). \tag{7.49}$$

Define an endomorphism A on $\mathbb{R}^d \times \mathbb{R}^d$ by $A(N_1, N_2) = (0, R_Q^{-2} N_1 + t_0\, S\, N_2)$. Then we obtain for the standard Euclidean scalar product, say $\langle \cdot, \cdot \rangle$, on $\mathbb{R}^d \times \mathbb{R}^d$, writing $N = (N_1, N_2)$,

$$R_Q^{-2} \langle N_1, N_2 \rangle = \langle A\,N, N \rangle - t_0\,\langle S\,N_2, N_2 \rangle. \tag{7.50}$$

Note that for (N_1, N_2) of (7.49), we have $\langle S N_2, N_2\rangle = Q[m]$ and $A[N] \stackrel{\text{def}}{=} \langle AN, N\rangle = \langle m, n\rangle$. Furthermore, note that the Euclidean norm on $\mathbb{R}^d \times \mathbb{R}^d$ induces a norm on the space L_d by restriction and that L_d is isometric to $\mathbb{R}^d$ (with the Euclidean norm) under an appropriately chosen isomorphism.

Meyer (1884), [Mey84], showed that there always exists for $d \geq 5$ a solution $N_0 \in \Lambda \setminus \{0\}$ of $A[N_0] = 0$, see Theorem 1.3. By the above mentioned isometry, the result of (Birch and Davenport (1958a)), [BD58b], see Theorem 1.3, applies to the integer valued quadratic form A on the lattice $\Lambda \stackrel{\text{def}}{=} R_Q\,(L_d \cap \Lambda_{t_0}) \subset L_d$. Hence there exist a solution $N \in \Lambda \setminus \{0\}$ of $A[N_0] = 0$ such that for an absolute constant c_d depending on the dimension d only

$$\|N\|^2 \leq c_d \Big(\operatorname{Tr} A^2\Big)^{(d-1)/2} \det(\Lambda)^2. \tag{7.51}$$

In view of (7.50) and (7.51) we obtain for this element $N = (N_1, N_2) \in \Lambda$ (of type (7.49)) by Cauchy–Schwartz and (7.51) together with (7.48)

$$\begin{aligned} |t_0 R_Q^2 Q[n]| \quad &\leq \quad \|N_1\|\,\|N_2\| \leq \|N\|^2 \leq c_d \Big(\operatorname{Tr} A^2\Big)^{(d-1)/2} \det(\Lambda)^2 \\ &\leq c_d \Big(\operatorname{Tr} A^2\Big)^{(d-1)/2} D_Q^2 \leq c_d\, d^{d-1}\, |t_0|^{d-1}\, D_Q^2, \end{aligned} \tag{7.52}$$

since $\operatorname{Tr} A^2 = t_0^2 \operatorname{Tr} S^2 = t_0^2 d$. Thus $|Q[m]| < 1$, provided that

$$|t_0 R_Q^2| > c_d\, d^{d-1}\, |t_0|^{d-1}\, D_Q^2. \tag{7.53}$$

Since $|t_0| \leq T_Q \ll_d \log q$, this requires $R_Q > c(d)\,(\log q)^{d/2-1}\, D_Q$, which holds by our choice (7.43) of R_Q, since $D_Q = c(d)\,|\det Q|^{1/2}\, q^{1/(1/2-\beta)}$. Furthermore, by (7.48), (7.49), (7.51) and (7.53) we conclude that

$$Q_+[n] \leq \|N\|^2 \ll_d |\det Q| q^{2/(1/2-\beta)} (\log q)^{d-1}, \tag{7.54}$$

thus proving the bound (1.10) in Theorem 1.4. Although $N \neq 0$, we still have to check that its component $N_2 = Q_+^{1/2} n$ is nonzero. Assume that $n = 0$ and hence $m \neq 0$. Then (7.49) implies that $N_2 = R_Q^2 Q_+^{1/2} m$. Hence by (7.54), $R_Q^2 q_0 \|m\|^2 \leq \|N\|^2 \ll_d D_Q^2 (\log q)^{d-1}$, which results in a contradiction for $R_Q = c(d)(\log q)^{(d-1)/2} D_Q$. This conludes the proof of Theorem 1.4.

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